Expected Utility Calibration for Continuous Distributions

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ABSTRACT

Recently, Rabin criticized the use of diminishing marginal utility in explaining risk aversion in small gambles with a mathematical theorem, which compares revealed risk averting behavior in small gambles to the risk behavior implied by expected utility theory in somewhat larger gambles, using discrete payoff distributions. To examine whether his criticism holds in more realistic risky situations, we generalize his theorem to the cases of continuous distributions and of continuous choice. The results suggest that the absolute size of the risk may not be as important as the relative size of the possible risk reduction, and that expected utility is likely a poor explanation for any short term risk response. We discuss some rules of thumb for judging the appropriateness of expected utility in practice.

Almost three centuries ago, Bernoulli arrived at the idea that people might choose among gambles not based on the expected value of outcomes but on the expected value of utility associated with the outcomes. With a formal axiomatization by von Neumann and Morgenstern in 1947, Expected Utility Theory (EUT) became the standard model of decision-making under uncertainty. Its inception was soon followed by a steady stream of empirical studies beginning with Allais' famous paradox in 1953, which revealed a variety of patterns in choice behavior that appear inconsistent with EUT. In addition to violations of the independence axiom, so-called common consequence effects and common ratio effects, observations of preference reversal (Lichtenstein and Slovic, 1971) and choices affected by framing (Tversky and Kahneman, 1981) challenged the theory's more fundamental assumptions of procedure and description invariance. A swarm of new, so-called non-expected utility theories emerged (see Starmer for a review), but EUT remains the dominant theory of choice under uncertainty in empirical economic research.

In EUT, risk aversion is explained completely by the concavity of the utility function. Friedman and Savage (1948) expanded the idea to represent various risk preferences with utility functions of different shapes, which broadened the appeal of EUT. Commonly used measures of absolute and relative risk aversion coefficients (Arrow, 1971) are direct properties of the utility curve. Yet, diminishing marginal utility of wealth (DMUW) can be justified by reasons other than risk aversion, the most important of which is the saturation effect where utility from additional units of the same good is smaller than that from the first. Hansson (1988) cautions that concavity of the utility function may have nothing to do with risk aversion.

Recently, Rabin (2000) illustrated this problem formally as a theorem. His calibration theorem shows that if an EU-maximizing decision-maker is averse to one small-stake risk of discrete nature at every wealth level, she must absurdly be averse to large risks. For example, a person who always turns down a lottery with a 0.5 probability of winning $125 and 0.5 probability of losing $100, will always turn down any bet with a 0.5 chance of losing $600, no matter how large a gain may be had with the remaining 0.5 probability. Neilson (2001) generalized Rabin's theorem to rank-dependent EUT, showing that newer theories exaggerate the problem because of the shape of probability weighting functions commonly found empirically.

The recent development in the descriptive theory of decision-making under uncertainty has led to little change within applied fields (with the possible exception of finance). We speculate that the main reason for this is the fact that counter-EUT examples have mostly involved dichotomous choice among discrete lotteries with small monetary stakes, which are not easily associated with many real-world applications. Furthermore, many of the arguments, including Rabin's theorem, employ the concepts of small-, moderate-, and large-stake risks, which are vague. Hence, applied economists have had no means to assess whether or not EUT is applicable in their particular case.[1]

In order to bridge this gap, we generalize the work of Rabin to the case of continuous distributions to examine choices between random outcomes of more reasonable size and scope, and outline steps that applied economists can use to evaluate how reasonable EUT may be in a particular application. We link our results to Arrow's risk aversion coefficient to discuss the size of stakes in a familiar framework. In order to broaden the applicability of our contribution beyond dichotomous choice, we illustrate a parallel analysis for a portfolio problem. By allowing economists to determine whether EUT is internally consistent over the range of choices of a decision-maker, we hope that many inappropriate applications may be eliminated.

[1] Neilson and Winter (2002) calibrate a single utility function to wage-fatality risk and portfolio choice data, to illustrate that a single utility function cannot explain the two risks.

In the next section, we present the calibration theorem for continuous distributions. Then, we illustrate the procedure that can be used to determine the applicability of EUT through an example using the uniform distribution. We relate our calibration results to Arrow's absolute and relative risk aversion coefficients to examine the scope of risk, in terms of wealth at risk consistent with EUT. Lastly, we extend our analysis to a continuous choice setting of portfolio choice.

We find that DMUW is a reasonable explanation for risk behavior in only two situations: (1) when the difference in means is very small, and (2) when the amount of wealth at reasonable risk is very large. Both of these cases may present extreme difficulty in empirical applications. If the difference in means between the relevant choices is very small, statistical tools cannot differentiate between the means. In this case it is difficult to determine if the individual is displaying DMUW, or is simply maximizing expected payout. On the other hand, very few real-world (irreversible) choices place enough wealth at risk to justify even a slight decline in mean payout.

General Expected Utility Calibration Theorem

An expected utility maximizer has a preference function over probability distributions $V(F)=\int U(w+z)dF(z)$, where U is the utility function defined over wealth levels, w is initial wealth, and z is the random change in wealth. Rabin, seeking to show the inconsistency of EUT in small and large gambles, assumes a utility function that is everywhere concave and exploits the continuity of the utility function. If the utility function is everywhere concave and continuous, then EUT implies that the individual must behave as if approximately risk neutral in small, nearly fair gambles. When individuals are observed to behave as if they are risk averse in these gambles, the level of concavity necessary to reconcile the behavior is beyond reason. The corollary to his calibration theorem is stated as follows:

Corollary 1 (Rabin, 2000): *Suppose that for all* $w, U'(w) > 0$, *and* $U''(w) < 0$. *Suppose there exists* $g > l > 0$, *such that, for all* w, $.5U(w-l) + .5U(w+g) < U(w)$. *Then for all positive integers* k, *and all* $m < m(k)$, $.5U(w-2kl) + .5U(w+mg) < U(w)$, *where*

$$m(k) = \begin{cases} \dfrac{\ln\left[1-\left(1-\dfrac{l}{g}\right)2\sum_{i=1}^{k}\left(\dfrac{g}{l}\right)^{i}\right]}{\ln\left(\dfrac{l}{g}\right)} - 1 & if \quad 1-\left(1-\dfrac{l}{g}\right)2\sum_{i=1}^{k}\left(\dfrac{g}{l}\right)^{i} > 0 \\ \infty & if \quad 1-\left(1-\dfrac{l}{g}\right)2\sum_{i=1}^{k}\left(\dfrac{g}{l}\right)^{i} \leq 0. \end{cases}$$

For a statement of the theorem and a proof of both the theorem and the corollary presented above, see the appendix of Rabin (2000). This corollary allows us to compare risk behavior over even-chance bets under the assumption of risk aversion. For example, a person who will always turn down a lottery with a 0.5 probability of winning \$110 and 0.5 probability of losing \$100—hereafter denoted by (0.5, 110, 0.5, -100)—will also always turn down the lottery represented by (0.5, 2930, 0.5, -800). Since EUT assumes that concavity is the only explanation of risk attitudes, outrageous behavior is implied unless the utility function changes from concave to convex as prizes become larger. It is easily confirmed by this collorary that if a global risk averter turns down (0.5, 125, 0.5, -100), they will always turn down any bet with a 0.5 chance of losing \$600, no matter how large of a gain may be had with the remaining 0.5 probability.

The approach of Rabin's theorem is to use the constraints imposed by concavity to determine the risk behavior implied by a particular choice in a dichotomous lottery case. In generalizing this process, we also rely heavily on revealed preferences and assume a utility function that is everywhere concave, allowing only concavity to explain choices. We wish to calibrate the flattest utility function to an observed risk averse decision and examine its implications for other choice behavior. Since less concavity implies a lower degree of risk aversion according to EUT, we can claim that DMUW cannot account for the observed choices under uncertainty if we find an absurd implication from this flattest utility representation. Our problem can be formulated mathematically in several ways, but below, we find the utility function with minimum concavity over the range of the support that justifies some observed risk-averse decision. An alternate method finds the utility function that minimizes the decline in its marginal value. These two approaches both yield similar results (see the appendix for a discussion of minimum decline).

Suppose an individual chooses the lottery represented by the probability density $f(x)$ when $g(x)$ is available where $\int_{\underline{x}}^{\overline{x}} xf(x)dx < \int_{\underline{x}}^{\overline{x}} xg(x)dx$. Then we wish to find $U(\cdot)$ that solves

$$\min_{\{U(x)\}}\left[\lim_{x\downarrow\underline{x}}\frac{U(x)-U(\underline{x})}{x-\underline{x}}-\lim_{x\uparrow\overline{x}}\frac{U(\overline{x})-U(x)}{\overline{x}-x}\right],$$

subject to

(1) $$U(\underline{x})=\underline{x},$$

(2) $$U(\overline{x})=\overline{x},$$

(3) $$\int_{\underline{x}}^{\overline{x}} U(x)f(x)dx \geq \int_{\underline{x}}^{\overline{x}} U(x)g(x)dx,$$

(4) $$U(x+t)-U(x)\leq U(x)-U(x-t), \qquad \forall x,t.$$

Here we have assumed that the support of x is $[\underline{x},\overline{x}]$ (or in the least that the two distributions do not differ outside of this range). Without loss of generality, we have set $U(\underline{x})=\underline{x}$, and

$U(\overline{x}) = \overline{x}$. The condition in (3) is the revealed preference condition, and (4) requires the function to be weakly concave (but not necessarily differentiable at any point). The use of left and right hand derivatives eliminate the need to define the utility function outside of the support. We will call this the revealed minimum concavity problem (RMCP).

Proposition: *The solution to the RMCP can be represented as two line segments, with*

(5) $$U_{(x^*,\beta)}(x) = \begin{cases} \overline{x} - \beta(\overline{x} - x^*) - \dfrac{\overline{x} - \underline{x} - \beta(\overline{x} - x^*)}{x^* - \underline{x}}(x^* - x) & if \quad x < x^* \\ \overline{x} - \beta(\overline{x} - x) & if \quad x \geq x^*, \end{cases}$$

where $\beta \leq 1$, *and* $\int_{\underline{x}}^{\overline{x}} U_{(x^*,\beta)}(x) f(x) dx = \int_{\underline{x}}^{\overline{x}} U_{(x^*,\beta)}(x) g(x) dx$.

Proof: Let U be any function that claims to satisfy the RMCP. In order for the condition in (1) to be satisfied, there must exist $\{\underset{\sim}{x}, \tilde{x}\} \in [\underline{x}, \overline{x}]$ with $f(x) \geq g(x)$ whenever $x \in [\underset{\sim}{x}, \tilde{x}]$.

(i) Suppose $g(x) > f(x)$ whenever $x \notin [\underset{\sim}{x}, \tilde{x}]$. Then define

$$\hat{U}(x) = \begin{cases} \underline{x} + \dfrac{U(\underset{\sim}{x}) - \underline{x}}{\underset{\sim}{x} - \underline{x}} x & \text{if} \quad x \leq \hat{x} \\ \overline{x} - \dfrac{\overline{x} - U(\tilde{x})}{\overline{x} - \tilde{x}}(\overline{x} - x) & \text{if} \quad x > \hat{x} \end{cases}$$

where $\underline{x} + \dfrac{U(\underset{\sim}{x}) - \underline{x}}{\underset{\sim}{x} - \underline{x}} \hat{x} = \overline{x} - \dfrac{\overline{x} - U(\tilde{x})}{\overline{x} - \tilde{x}}(\overline{x} - \hat{x})$.

Then, $\hat{U}(x) \geq U(x)$ whenever $x \in [\underset{\sim}{x}, \tilde{x}]$, and $\hat{U}(x) \leq U(x)$ whenever $x \notin [\underset{\sim}{x}, \tilde{x}]$. Hence,

$\int_{\underline{x}}^{\overline{x}} \hat{U}(x)[f(x) - g(x)]dx \geq \int_{\underline{x}}^{\overline{x}} U(x)[f(x) - g(x)]dx \geq 0$. Also, $\dfrac{\overline{x} - \hat{U}(\tilde{x})}{\overline{x} - \tilde{x}} > U'(\overline{x})$, and

$\dfrac{\underline{x} - \hat{U}(\underset{\sim}{x})}{\underline{x} - \underset{\sim}{x}} < U'(\underline{x})$ by concavity, hence U cannot solve the RMCP and we have a contradiction.

(ii) Analogous to the proof in (i), if the densities cross multiple times between $\underline{x}$ and $\overline{x}$, we can define a line segment for each range of x where $g(x) > f(x)$. For example, with two such reversals $[\underset{\sim}{x}_1, \tilde{x}_1]$ and $[\underset{\sim}{x}_2, \tilde{x}_2]$ with $\tilde{x}_1 < \underset{\sim}{x}_2$, define $\hat{U}$ as:

$$\hat{U}(x) = \begin{cases} \underline{x} + \dfrac{U(\underset{\sim}{x}_1) - \underline{x}}{\underset{\sim}{x}_1 - \underline{x}}(x - \underline{x}) & \text{if} \quad x \leq \hat{x}_1 \\ U(\tilde{x}_1) - \dfrac{U(\underset{\sim}{x}_2) - U(\tilde{x}_1)}{\underset{\sim}{x}_2 - \tilde{x}_1}(x - \tilde{x}_1) & \text{if} \quad \hat{x}_1 < x \leq \hat{x}_2 \\ \overline{x} - \dfrac{\overline{x} - U(\tilde{x}_2)}{\overline{x} - \tilde{x}_2}(\overline{x} - x) & \text{if} \quad \hat{x}_2 < x \end{cases}$$

where $\underline{x} + \dfrac{U(\underset{\sim}{x}_1) - \underline{x}}{\underset{\sim}{x}_1 - \underline{x}}(\hat{x}_1 - \underline{x}) = U(\tilde{x}_1) + \dfrac{U(\underset{\sim}{x}_2) - U(\tilde{x}_1)}{\underset{\sim}{x}_2 - \tilde{x}_1}(\hat{x}_1 - \tilde{x}_1)$ and

$U(\tilde{x}_1) + \dfrac{U(\underset{\sim}{x}_2) - U(\tilde{x}_1)}{\underset{\sim}{x}_2 - \tilde{x}_1}(\hat{x}_2 - \tilde{x}_1) = \overline{x} - \dfrac{\overline{x} - U(\tilde{x}_2)}{\overline{x} - \tilde{x}_2}(\overline{x} - \hat{x}_2)$.

Again, $\int_{\underline{x}}^{\overline{x}} \hat{U}(x)[f(x) - g(x)]dx \geq \int_{\underline{x}}^{\overline{x}} U(x)[f(x) - g(x)]dx \geq 0$. Also, $\dfrac{\overline{x} - \hat{U}(\tilde{x}_2)}{\overline{x} - \tilde{x}_2} > U'(\overline{x})$, and

$\dfrac{\underline{x} - \hat{U}(\underset{\sim}{x}_1)}{\underline{x} - \underset{\sim}{x}_1} < U'(\underline{x})$ by concavity, hence U cannot solve the RMCP and we have a contradiction.

(iii) Further, suppose that we have determined that the solution to the RMCP must consist of $n > 2$ connected line segments. Then the RMCP can be restated as

(6)
$$\min_{\{\beta_2, \ldots, \beta_n, \hat{x}_1, \ldots, \hat{x}_{n-1}\}} \frac{\overline{x} - \beta_n(\overline{x} - \hat{x}_{n-1}) - \sum_{i=2}^{n-1} \beta_i(\hat{x}_i - \hat{x}_{i-1}) - \underline{x}}{\hat{x}_1 - \underline{x}} - \beta_n$$

subject to

(7)
$$\beta_j \geq \beta_i \qquad \forall j < i,$$

$$(8)\qquad \frac{\overline{x}-\beta_n\left(\overline{x}-\hat{x}_{n-1}\right)-\sum_{i=2}^{n-1}\beta_i\left(\hat{x}_i-\hat{x}_{i-1}\right)-\underline{x}}{\hat{x}_1-\underline{x}}\geq\beta_2,$$

$$(9)\qquad \hat{x}_i\geq\hat{x}_j\qquad\forall i>j,$$

$$(10)\qquad \begin{aligned}&\int_{\hat{x}_{n-1}}^{\overline{x}}\left[\beta_n\left(\overline{x}-x\right)\right]\left[f(x)-g(x)\right]dx+\\&\sum_{i=2}^{n-1}\int_{\hat{x}_{i-1}}^{\hat{x}_i}\left[\left(\overline{x}-\beta_n\left(\overline{x}-\hat{x}_{n-1}\right)-\sum_{j=i}^{n-1}\beta_j\left(\hat{x}_j-\hat{x}_{j-1}\right)\right)+\beta_i\left(x-\hat{x}_{i-1}\right)\right]\left[f(x)-g(x)\right]dx+\\&\int_{\underline{x}}^{\hat{x}_1}\left[\underline{x}+\frac{\overline{x}-\beta_n\left(\overline{x}-\hat{x}_{n-1}\right)-\sum_{i=2}^{n-1}\beta_i\left(\hat{x}_i-\hat{x}_{i-1}\right)-\underline{x}}{\hat{x}_1-\underline{x}}\left(x-\underline{x}\right)\right]\left[f(x)-g(x)\right]dx\geq 0.\end{aligned}$$

where $\beta_i=\dfrac{U(\hat{x}_i)-U(\hat{x}_{i-1})}{\hat{x}_i-\hat{x}_{i-1}}$ is the slope of the utility curve in the segment $[\hat{x}_{i-1},\hat{x}_i]$, $i = 2,\ldots,n$.

Because we have assumed the minimum number of line segments, we know that (9) cannot bind. Suppose we knew the values for $\hat{x}_1,\ldots,\hat{x}_{n-1}$ that solve the above problem. Then the solution to the RMCP must solve (6) subject to (7), (8), and (10). Note that this problem is now linear in constraints. Because none of the constraints listed in (7), (8), and (10) have the same gradient as the maximand, we know that at least two of these constraints must bind by the principle of the simplex algorithm. But this means that two consecutive line segments have the same slope, and, hence, we could have eliminated one more line segment. This is a contradiction. Thus, the solution can be represented as (5) above.

■

The proof above allows us to narrow our consideration of utility functions, focusing on the functions that can be represented as (5), and satisfying $\int_{\underline{x}}^{\overline{x}}\left[g(x)-f(x)\right]U_{(x^*,\beta)}(x)dx=0$. This constraint can be solved for β. Substituting (5) into the equality yields

$$\int_{\underline{x}}^{x^*}\left[\overline{x}-\beta\left(\overline{x}-x^*\right)-\frac{\overline{x}-\underline{x}-\beta\left(\overline{x}-x^*\right)}{x^*-\underline{x}}\left(x^*-x\right)\right]\left[g(x)-f(x)\right]dx+$$
$$\int_{x^*}^{\overline{x}}\left[\overline{x}-\beta\left(\overline{x}-x\right)\right]\left[g(x)-f(x)\right]dx=0,$$

or,

$$(11)\ \beta=\frac{\int_{\underline{x}}^{x^*}\left[\underline{x}x^*+\overline{x}x-\overline{x}\underline{x}-\underline{x}x\right]\left[g(x)-f(x)\right]dx+\int_{x^*}^{\overline{x}}\left[\overline{x}x^*-\overline{x}\underline{x}\right]\left[g(x)-f(x)\right]dx}{\int_{\underline{x}}^{x^*}\left[\overline{x}x-x^*x+x^*\underline{x}-\overline{x}\underline{x}\right]\left[g(x)-f(x)\right]dx+\int_{x^*}^{\overline{x}}\left[\left(x^*-\underline{x}\right)\left(\overline{x}-x\right)\right]\left[g(x)-f(x)\right]dx}.$$

Then, the objective of the RMCP is to minimize:

$$(12)\qquad \frac{\overline{x}-\underline{x}-\beta\left(\overline{x}-x^*\right)}{x^*-\underline{x}}-\beta=\frac{\left(\overline{x}-\underline{x}\right)\left(1-\beta\right)}{x^*-\underline{x}},$$

or, substituting (11) for β,

$$(13)\qquad \frac{-\left(\overline{x}-\underline{x}\right)\int_{\underline{x}}^{\overline{x}}x\phi(x)dx}{\left(\left(x^*-\overline{x}\right)\int_{\underline{x}}^{x^*}\left(\underline{x}-x\right)\phi(x)dx+\left(x^*-\underline{x}\right)\int_{x^*}^{\overline{x}}\left(\overline{x}-x\right)\phi(x)dx\right)}.$$

Optimization of (13) requires only that we solve the standard first order and second order conditions. The first order condition can be represented as

$$\frac{(\overline{x}-\underline{x})\int_{\underline{x}}^{\overline{x}} x\phi(x)dx\left(\underline{x}\int_{\underline{x}}^{x^*}\phi(x)dx+\overline{x}\int_{x^*}^{\overline{x}}\phi(x)dx-\int_{\underline{x}}^{\overline{x}}x\phi(x)dx\right)}{\left((x^*-\overline{x})\int_{\underline{x}}^{x^*}(\underline{x}-x)\phi(x)dx+(x^*-\underline{x})\int_{x^*}^{\overline{x}}(\overline{x}-x)\phi(x)dx\right)^2}=0$$

where $\phi(x)=g(x)-f(x)$. Since the first integral in the numerator must be positive (by assumption), the first order condition can be represented by

(14) $$\underline{x}\int_{\underline{x}}^{x^*}\phi(x)dx+\overline{x}\int_{x^*}^{\overline{x}}\phi(x)dx=\int_{\underline{x}}^{\overline{x}}x\phi(x)dx.$$

Because $\int_{\underline{x}}^{x^*}\phi(x)dx=-\int_{x^*}^{\overline{x}}\phi(x)dx$, we can rewrite (14) as

(15) $$(\underline{x}-\overline{x})\int_{\underline{x}}^{x^*}\phi(x)dx=\int_{\underline{x}}^{\overline{x}}x\phi(x)dx.$$

The second order condition simplifies to

$$\left(\frac{2\left[(\overline{x}-\underline{x})\int_{\underline{x}}^{x^*}\phi(x)dx+\int_{\underline{x}}^{\overline{x}}x\phi(x)dx\right]^2}{\left[x^*(\overline{x}-\underline{x})\int_{\underline{x}}^{x^*}\phi(x)dx-(\overline{x}-\underline{x})\int_{\underline{x}}^{x^*}x\phi(x)dx+(x^*-\underline{x})\int_{\underline{x}}^{\overline{x}}x\phi(x)dx\right]}-\phi(x^*)(\overline{x}-\underline{x})\right)>0.$$

The limits of integration in these conditions can be extended (e.g., to the whole real line), so long as it is known that the kink in the utility function occurs on $x\in[\underline{x},\overline{x}]$.

Calibration Exercise in Practice

In this section we illustrate how our calibration problem may be used to determine if EUT is a reasonable explanation of an applied problem. We propose three steps:

1. Estimation of the wealth uncertainty for all relevant choices.
2. Determination of the minimum concavity utility function explaining the observed choice.

3. Comparison of the level of concavity observed over the interval of risk with the level of wealth.

Step 1 should be feasible in most applications. By using maximum entropy (ME) estimation (Zellner 1997), or some other appropriate estimation technique, a reasonable approximation of the distribution may be obtained. ME estimation may be particularly appropriate, since it exaggerates the diffusion of a distribution (thus biasing results away from a finding that EUT is not reasonable) and yields results even with very little data. Step 2 applies the general calibration theorem to the specific case. Step 3 involves calculating the ratio of value of a dollar at the supremum of the support to the value of a dollar at the infemum necessary for EU to explain the decision. We illustrate our proposal in an example using a uniform distribution for simplicity. Use of the uniform distribution eliminates the need for use of a numerical solver for the first order condition.

Suppose an individual is faced with the following distributions of payoffs:

$$g(x) = \begin{cases} \dfrac{1}{D_g} & \text{if} \quad x_g - \dfrac{D_g}{2} < x < x_g + \dfrac{D_g}{2} \\ 0 & \text{otherwise,} \end{cases}$$

$$f(x) = \begin{cases} \dfrac{1}{\alpha D_g} & \text{if} \quad x_g - \delta D_g - \dfrac{\alpha D_g}{2} < x < x_g - \delta D_g + \dfrac{\alpha D_g}{2} \\ 0 & \text{otherwise,} \end{cases}$$

where $D_g > 0$, x_g, δ, and α are parameters. We examine the case where $\delta \geq 0, \alpha \in [0,1)$. Note, (i) $\int xf(x)dx = \int xg(x)dx - 2\sqrt{3}\delta\sigma_g$, and (ii) $\sigma_f = \alpha\sigma_g$, where σ_f and σ_g are standard deviations of lotteries *f* and *g*, respectively. Hence, the parameter δ measures the decrease in expected value as a fraction of variation given by g, and α measures the variation implied by f as a fraction of variation given by g.

Then, $\phi(x) = g(x) - f(x)$ is defined as:

$$\phi(x) = \begin{cases} \dfrac{\alpha-1}{\alpha D_g} & \text{if} \quad x_g - \delta D_g - \dfrac{\alpha D_g}{2} < x < x_g - \delta D_g + \dfrac{\alpha D_g}{2} \\ \dfrac{1}{D_g} & \text{if} \quad x \in \left[x_g - \dfrac{D_g}{2}, x_g - \delta D_g - \dfrac{\alpha D_g}{2} \right] \cup \left[x_g - \delta D_g + \dfrac{\alpha D_g}{2}, x_g + \dfrac{D_g}{2} \right] \\ 0 & \text{if} \quad x \notin \left[x_g - \dfrac{D_g}{2}, x_g + \dfrac{D_g}{2} \right] \end{cases}$$

Hence, the solution to the RMCP can be found by substituting into (13),

$$-\int_{b_1}^{b_2} dx - \int_{b_2}^{x^*} \frac{\alpha-1}{\alpha} dx = \delta D_g \ .$$

where $b_1 = x_g - \dfrac{D_g}{2}$, and $b_2 = x_g - \delta D_g - \dfrac{\alpha D_g}{2}$. Solving this for x^* yields

$$x^* = x_g - \delta D_g \ .$$

Substituting into (13) yields

$$\beta - \frac{x_g + \dfrac{D_g}{2} - \beta\left(x_g + \dfrac{D_g}{2} - x^* \right) - x_g + \dfrac{D_g}{2}}{x^* - x_g + \dfrac{D_g}{2}} = \frac{8\delta}{\alpha - 1 + 4\delta^2} .$$

Let γ be the ratio between the upper and lower slopes of the solution of RMCP, which in our example is:

$$\text{(16)} \qquad \gamma = \frac{\beta}{\left(\dfrac{x_g + \dfrac{D_g}{2} - \beta\left(x_g + \dfrac{D_g}{2} - x^* \right) - x_g + \dfrac{D_g}{2}}{x^* - x_g + \dfrac{D_g}{2}} \right)} = \frac{1 - \alpha - 4\delta + 4\delta^2}{1 - \alpha + 4\delta + 4\delta^2} .$$

The parameter γ represents the ratio of the utilities of a dollar at the supremum and infemum of the support under the smallest possible decline in the marginal utility of wealth. Table 1 reports values of γ for various values of α and δ. The value of $2\sqrt{3}\delta$ is reported, since the mean of the preferred lottery is discounted by δD_g, which is equal to $2\sqrt{3}\delta$ times the standard deviation of the non-preferred lottery. Similarly, the standard deviation of the preferred lottery is $1-\alpha$ percent lower than the standard deviation of the non-preferred lottery. The mark "-" indicates the ratio is negative, or that the action cannot be rationalized with a monotonically increasing utility function.

Table 1 shows that an individual who faces a wealth prospect between $\underline{N}$ and $\overline{N}$, and is willing to trade a reduction in mean equal to one tenth of the standard deviation for a nine tenths reduction in standard deviation (i.e., $\delta 2\sqrt{3} = .1$ and $1-\alpha = .9$) values the $\overline{N}$ th dollar at most 77 percent as much as the $\underline{N}$ th dollar. This on its own seems extreme, but may be a reasonable assumption if the support of the distribution is sufficiently large. Other values would appear to be ridiculous even for large supports. In the very least, it must be agreed that a negative marginal value of wealth cannot be tolerated over any interval (at least those observed in practice), and EUT is not applicable in such cases. Thus, empirical researchers employing EUT must first determine if the degree of DMUW is reasonable for their particular application by computing the values of γ for each observed choice. If the implied decline in marginal utility is unreasonable for the particular risks under consideration, the researcher should seek other explanations for the risk behavior.

Rabin showed that risk aversion in small gambles leads to absurd results. Our results in Table 1 shows that absurd results are obtained when individuals give up nearly any amount in mean for a small reduction in standard deviation. For example, by reducing the standard deviation by nine tenths, the dollar must lose 40 percent its marginal value over the support if a discount equal to two tenths of the standard deviation is taken (i.e., $\delta 2\sqrt{3} = .2$ and $1-\alpha = .9$). This illustrates that Rabin's concavity problem is much larger than originally supposed and must be addressed in practical applications. To consider the magnitude in a more identifiable context, we next consider calibration in terms of common measures of risk aversion.

Arrow's Risk Aversion Coefficients and Calibration

Arrow (1971) proposed the following measure of relative risk aversion in terms of first and second derivatives of the utility function and wealth:

$$R_R = -\frac{U''(w)}{U'(w)} w,$$

which is invariant with respect to changes in the units of utility and wealth. It can be shown mathematically that R_R cannot approach a limit above one as wealth approaches zero and below one as wealth approaches infinity if the utility function is bounded.

In the case of our RMCP utility function, an analog to Arrow's measure may be found by dividing the objective function from (12) by $(\overline{x} - \underline{x})$, or

$$R_R \approx -\frac{U'(\overline{x}) - U'(\underline{x})}{U(\overline{x}) - U(\underline{x})} x = \frac{1-\beta}{x^* - \underline{x}} x.$$

For the example above we find

$$R_R = \frac{8\delta}{\left(1-\alpha-4\delta^2\right)} \frac{x}{D_g}$$

If this coefficient is to be near 1, then it must be that

(17) $$D_g \approx D_g R_R = \frac{8\delta}{\left(1-\alpha-4\delta^2\right)} x = D_g R_A x.$$

Since D_g is the support of the risky prospects, which are assumed to be uniformly distributed, and x is the wealth level, R_A may reasonably be interpreted as the portion of wealth that must be at risk for the individual's relative risk aversion coefficient to be near 1.

In Table 2, the value of R_A is given for various values of α and δ. For example, if about a quarter of wealth is at risk (R_A = 0.26), an individual would give up one tenth of a standard deviation in mean for a 90 percent reduction in variance, which may seem reasonable. If an individual is willing to exchange more than two tenth of a standard deviation in mean for any reduction in variance, more than half of her wealth must be at risk.

By comparing the range of particular risks to the wealth levels of individuals, it is possible to gauge whether risk behavior is reasonably due to DMUW, or whether other explanations need to be entertained. The numbers in Table 2 suggest that expected utility may be inappropriate as a model for most short term risks. Decisions that may be reversed within the short term are not likely to involve large portions of one's wealth, while a large decline in marginal utility over a short support may be reasonable if the individual has very little wealth.

Because estimation of risk aversion coefficients involves averaging over diverse behaviors, estimation may not reveal behavior that is absurd given EUT. A small portion of a sample behaving as-if risk neutral could obscure the fact that individuals in the sample could not be motivated by DMUW. If a significant portion of a sample are behaving in a way that is unreasonable given the assumptions of EUT, some other explanation must be found for risk response in the sample. Empirical researchers should compare the support to wealth level for each wealth cohort in a sample. If any cohort is behaving in a way inconsistent with EUT, using EUT to explain other cohorts is questionable.

Portfolio Analysis

Our analysis in the previous sections used dichotomous choice, where most of EUT violations have been reported in the past. We now extend our calibration efforts to continuous choice using portfolio selection problem to broaden the applicability of our results. Let us consider an example based on Mas-Colell, Whinston and Green (1995, p. 188). Suppose the individual can purchase shares of either a risky or a safe asset. Let the return from the risky asset be given by x, with distribution $h(x)$, for each dollar invested. The safe asset returns one dollar for every dollar invested. Hence, the optimization problem can be represented as

$$\max_{A,B} \int_{\underline{x}}^{\overline{x}} u(Ax+B)h(x)dx,$$

$$\text{subject to } A+B=w,$$

where A and B are dollars invested in risky and safe assets, respectively, and w is initial wealth. This problem will be solved by some bundle (A^*,B^*). However, the individual could have chosen any $(A^*+\varepsilon, B^*-\varepsilon)$ where $-A^*<\varepsilon<B^*$.[2] Define the distributions of payout from the revealed preferred portfolio as

$$f(y)=h\left(\frac{y-B^*}{A^*}\right)\frac{1}{A^*}.$$

Then, we can define the distribution of payout from all possible portfolios as

$$g(y)=h\left(\frac{y-(B^*-\varepsilon)}{A^*+\varepsilon}\right)\frac{1}{A^*+\varepsilon}.$$

For the sake of our example, let

[2] Note that if the prices differ, ε must be weighted by price.

$$h(x)=\begin{cases}\dfrac{1}{D_h} & \text{if} \quad x_h-\dfrac{D_h}{2}<x<x_h+\dfrac{D_h}{2}\\ 0 & \text{otherwise,}\end{cases}$$

where $x_h > 1$, so the average net return is positive. Then,

$$f(y)=\begin{cases}\dfrac{1}{A^*D_h} & \text{if} \quad A^*x_h-\dfrac{A^*D_h}{2}+B^*<y<A^*x_h+\dfrac{A^*D_h}{2}+B^*\\ 0 & \text{otherwise,}\end{cases}$$

and

$$g(y)=\begin{cases}\dfrac{1}{\left(A^*+\varepsilon\right)D_h} & \text{if} \quad \left(A^*+\varepsilon\right)x_h-\dfrac{\left(A^*+\varepsilon\right)D_h}{2}+B^*-\varepsilon<y<\left(A^*+\varepsilon\right)x_h+\dfrac{\left(A^*+\varepsilon\right)D_h}{2}+B^*-\varepsilon\\ 0 & \text{otherwise.}\end{cases}$$

If we wish to examine the gambles that would have resulted in greater expected value, we must restrict our attention to $\varepsilon > 0$. The choice of $\varepsilon = 0$ does not satisfy the assumptions of the RMCP, because the two choices compared are identical. This problem is now similar to the discrete choice example from the previous section, where

$$\alpha=\frac{A^*}{A^*+\varepsilon},$$

$$\delta=\frac{\varepsilon\rho}{A^*+\varepsilon},$$

where $\rho=\dfrac{x_h-1}{D_h}$.

We can find γ as a function of ε:

$$\gamma=\frac{A^*\left(1-4\rho\right)+\varepsilon\left(1-4\rho+4\rho^2\right)}{A^*\left(1+4\rho\right)+\varepsilon\left(1+4\rho+4\rho^2\right)}.$$

This is the maximized value for each possible ε that was not chosen. We can now find the minimum percent of diminishing marginal value of wealth implied by the revealed choice by solving

$$\min_{\varepsilon \in (0, B^*]} \gamma(\varepsilon).$$

Since, γ is increasing in ε, the solution is

$$\lim_{\varepsilon \downarrow 0} \gamma = \frac{1-4\rho}{1+4\rho}.$$

The values of γ for various ρ are reported in Table 3. Clearly this value will be negative if $\rho > 1/4$, i.e., the risky return is greater than 25 percent, in which case holding any of the safe asset is irrational given EUT. An alternative method shows that the revealed choice can be rationalized so long as $\rho < 1/2$ (see the appendix for a discussion). Thus, EUT implies that holding any of the safe asset when the risky return is greater than 50 percent cannot be rational. In this case, if DMUW is the only explanation for risky behavior, then the dollar has non-positive marginal value to the individual. Thus, EUT may not be well suited to explain risky portfolios in the presence of high return assets.

Note that D_h measures diffusion of one unit of A, and $A^* D_h$ is the total diffusion of the portfolio return. Suppose that $D_h = 1$ and $\rho = 0.1$, so $x_h = 1.1$, and the individual could lose at most 40 percent of her investment in asset A. Then, if $A^* = 100$, any holding of the safe asset reflects a utility function that values the 100th dollar invested in the risky asset equal to \$0.43 of the first dollar. With a larger investment in A^*, this diminishing of marginal utility may seem more reasonable.

To see what share of investment would make RMCP reasonable in this case, we can solve for the portion of wealth under risk that is consistent with the relative risk aversion coefficient near 1 (equation (17)):

$$D_g R_A x = D_h \left(A^* + \varepsilon\right) R_A x = \frac{8\left(A^* + \varepsilon\right)\rho}{A^* + \varepsilon + 4\varepsilon\rho^2} x .$$

The minimum diffusion of risk that is reasonable for each possible ε that was not chosen is:

$$D_h \approx \frac{8\rho}{A^* + \varepsilon + 4\varepsilon\rho^2} x .$$

To find the minimum diffusion of risk that can be justified for the overall revealed choice, we look for its maximum over all possible choices:

$$\max_{\varepsilon} \frac{8\rho}{A^* + \varepsilon + 4\varepsilon\rho^2} x .$$

Since the proportion of wealth at risk is decreasing in ε, the solution is

$$\lim_{\varepsilon \downarrow 0} \frac{8\rho}{A^* + \varepsilon + 4\varepsilon\rho^2} x = \frac{8\rho}{A^*} x = \frac{8\left(x_h - 1\right)}{A^* D_h} x .$$

Thus, the relationship between the wealth and risk diffusion that is consistent with revealed preferences is:

$$D_h \approx \frac{8\left(x_h - 1\right)}{A^* D_h} x .$$

As expected, the proportion of wealth that must be at risk is smaller when the expected return (variability) from a risky asset is higher (smaller).

Considering the example above ($D_h = 1, x_h = 1.1$), a reasonable level of wealth would be $x = \frac{5}{4} A^*$. Hence, one must be placing nearly 80 percent of their wealth in the risky asset for this decision to be consistent with EUT, and this must be risk that cannot be reversed by later decisions. For example, if we were studying stock market returns, an individual generally has an opportunity to sell their stock continuously during any business day. At any point in time, an individual's irreversible risk is the risk faced before they can execute another decision. While many individuals may have 80 percent of wealth tied up in investments, it is highly improbable (outside of Enron-style arrangements) that any would lose such a high percentage of wealth before they *could* sell in the next instant.

Conclusion

We have outlined a set of non-parametric tests to determine whether EUT is an appropriate explanation for observed behavior in applied settings. Violations of EUT found in the laboratory have long called empirical application into question. However, the lack of salient natural evidence against EUT and the peculiar nature of laboratory risks have led to the continued use of EUT in nearly all applied work. We believe that our proposed calibration tool may necessitate change in many of the current applied literatures on risk.

The calibration tool could be applied to any risk whether dichotomous or continuous choice. Further, as with Rabin's work, our calibration technique should generalize to the popular alternatives to EUT. Truly, Rabin's concavity problem is not due to risk aversion in the face of absolutely small risks. Rather, EUT will make ridiculous predictions when individuals behave risk aversely to small differences in risk. The possibility of observing the concavity problem in risks of reasonable size makes it relevant to applied researchers. We are currently working on employing calibration to risks faced by agricultural producers.

The calibrations of simple risks used in our examples suggest that DMUW is a poor explanation of risk response unless the risky choices have a very small difference in mean, or the irreversible risk is large relative to wealth. If the difference in means is small, there may be problems with estimation. In our example, EUT may be appropriate if the mean outcomes differ by less than two tenths of the standard deviation. This raises questions about whether statistical tools would be able to determine a significant difference in the means of any two distributions. Unless the differences in the variance of the two distributions are large, distinguishing between the means of the distributions would require large amounts of data. If econometricians have difficulty differentiating between means, individuals without statistical resources will likely have at least as much difficulty determining the higher average payoff. Even in the gambles where EUT makes the most sense, it is hard to reject the notion that behavior is driven by differences in perception of the means rather than the difference in variance.

We believe that many of the common applications of DMUW are not accurate portrayals of the actual risk behavior (for a debate on normative and positive aspects of EUT, see Watt versus Rabin and Thaler, 2002). Exposing the problems of EUT assumptions is a necessary step in expanding our understanding of this behavior. Where DMUW is found to be inadequate, other tools will need to be developed. Most alternatives to EUT still use DMUW as a primary explanation of risk response. It may be more fruitful to investigate the availability and use of information, problems in understanding, known psychological biases and other factors not currently considered by applied economists. Certainly we cannot model risk behavior if we do not know how individuals perceive the risks they face.

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Appendix: An Alternative to the Revealed Minimum Concavity Problem

Instead of minimizing concavity, we could search for a utility function with a minimum decline in marginal utility over the support. For such a function, the ratio between the marginal utilities at the supremum and infemum of the support, defined in (16) as γ, is maximized. Thus, the RMCP can alternatively be formulated as:

(18) $$\max_{\{U(x)\}} \frac{\left(\lim_{x \uparrow \bar{x}} \frac{U(\bar{x}) - U(x)}{\bar{x} - x}\right)}{\left(\lim_{x \downarrow \underline{x}} \frac{U(x) - U(\underline{x})}{x - \underline{x}}\right)}$$

subject to constraints (1) through (4). The solution to this problem is identical to that stated for the RMCP in the Proposition. Thus, this problem can be simplified to:

(19) $$\max_{x^*} \frac{x^* \int_{\underline{x}}^{x^*} \phi(x)dx - \int_{\underline{x}}^{x^*} x\phi(x)dx}{x^* \int_{\underline{x}}^{x^*} \phi(x)dx + \int_{x^*}^{\bar{x}} x\phi(x)dx}.$$

The first order condition for (19) reduces to

$$\int_{\underline{x}}^{x^*} \phi(x)dx = 0.$$

In the case of our dichotomous choice example, we find

$$\tilde{\gamma} = \frac{(1-\alpha-2\delta)^2}{(1-\alpha+2\delta)^2}.$$

Using this formula, we can rewrite Table 1 as Table 4. The values of γ and $\tilde{\gamma}$ are similar, particularly when α is small. The difference is

$$\tilde{\gamma} - \gamma = \frac{32\alpha\delta^3}{\left(1-\alpha+4\delta+4\delta^2\right)(\alpha-1-2\delta)^2}.$$

Values of $\tilde{\gamma}-\gamma$ are plotted in Figure 1. These values are less than 0.1 when $1-\alpha \leq 2\sqrt{3}\delta$ (when the discount in mean is less than the reduction in standard deviation), or, southeast of the 45 degree line. In the portfolio example, we find

$$\tilde{\gamma}=\frac{(1-2\rho)^2}{(1+2\rho)^2},$$

which implies a utility function that remains monotonic for a larger value of ρ than that derived from the RMCP. The choice of A^* and B^* can now be rationalized if $\rho<0.5$. However, $\tilde{\gamma}$ is unreasonably small for $0.25<\rho<0.5$. For example, when $\rho=0.25$, $\tilde{\gamma}=\frac{1}{9}$, implying that the last dollar invested in a risky asset is valued only one-ninth of the first dollar. This cannot be sensible for a typical investment amount. Hence, EUT is likely inappropriate in this case as well, although the revealed preference is rational.

Table 1. Values of γ for various values of α, δ .[a]

Percent Reduction in SD $1-\alpha=$	Mean Discount in Percent SD $\delta 2\sqrt{3}=$				
	0.1	**0.2**	**0.3**	**0.4**	**0.5**
0.9	**0.77**	**0.60**	**0.46**	**0.35**	**0.26**
0.8	**0.74**	**0.56**	**0.41**	**0.30**	**0.21**
0.7	**0.72**	**0.51**	**0.36**	**0.24**	**0.15**
0.6	**0.68**	**0.45**	**0.29**	**0.17**	**0.08**
0.5	**0.63**	**0.38**	**0.21**	**0.09**	**0.01**
0.4	**0.55**	**0.28**	**0.11**	**-**	**-**
0.3	**0.45**	**0.15**	**-**	**-**	**-**
0.2	**0.28**	**-**	**-**	**-**	**-**
0.1	**-**	**-**	**-**	**-**	**-**
0.0	**-**	**-**	**-**	**-**	**-**

[a] The discount of δD_g is equal to $2\sqrt{3}\delta$ times the standard deviation of $g(x)$. Similarly, $1-\alpha$ measures the percentage reduction in standard deviation of $g(x)$. The mark "-" indicates the maximum value is negative, or that the action cannot be rationalized with a monotonically increasing utility function.

Table 2. Values of R_A for various values of α, δ.[a]

Percent Reduction in SD $1-\alpha =$	Mean Discount in Percent SD $\delta 2\sqrt{3} =$				
	0.1	**0.2**	**0.3**	**0.4**	**0.5**
0.9	**0.26**	**0.52**	**0.80**	**1.09**	**1.41**
0.8	**0.29**	**0.59**	**0.90**	**1.24**	**1.61**
0.7	**0.33**	**0.67**	**1.03**	**1.43**	**1.87**
0.6	**0.39**	**0.79**	**1.22**	**1.69**	**2.23**
0.5	**0.46**	**0.95**	**1.47**	**2.07**	**2.77**
0.4	**0.58**	**1.19**	**1.87**	**-**	**-**
0.3	**0.78**	**1.61**	**-**	**-**	**-**
0.2	**1.17**	**-**	**-**	**-**	**-**
0.1	**-**	**-**	**-**	**-**	**-**
0.0	**-**	**-**	**-**	**-**	**-**

[a] The discount of δD_g is equal to $2\sqrt{3}\delta$ times the standard deviation of $g(x)$. Similarly, $1-\alpha$ measures the percentage reduction in standard deviation of $g(x)$. The mark “-“ indicates the maximum value is negative, or that the action cannot be rationalized with a monotonically increasing utility function.

Table 3. Values of γ for various ρ.

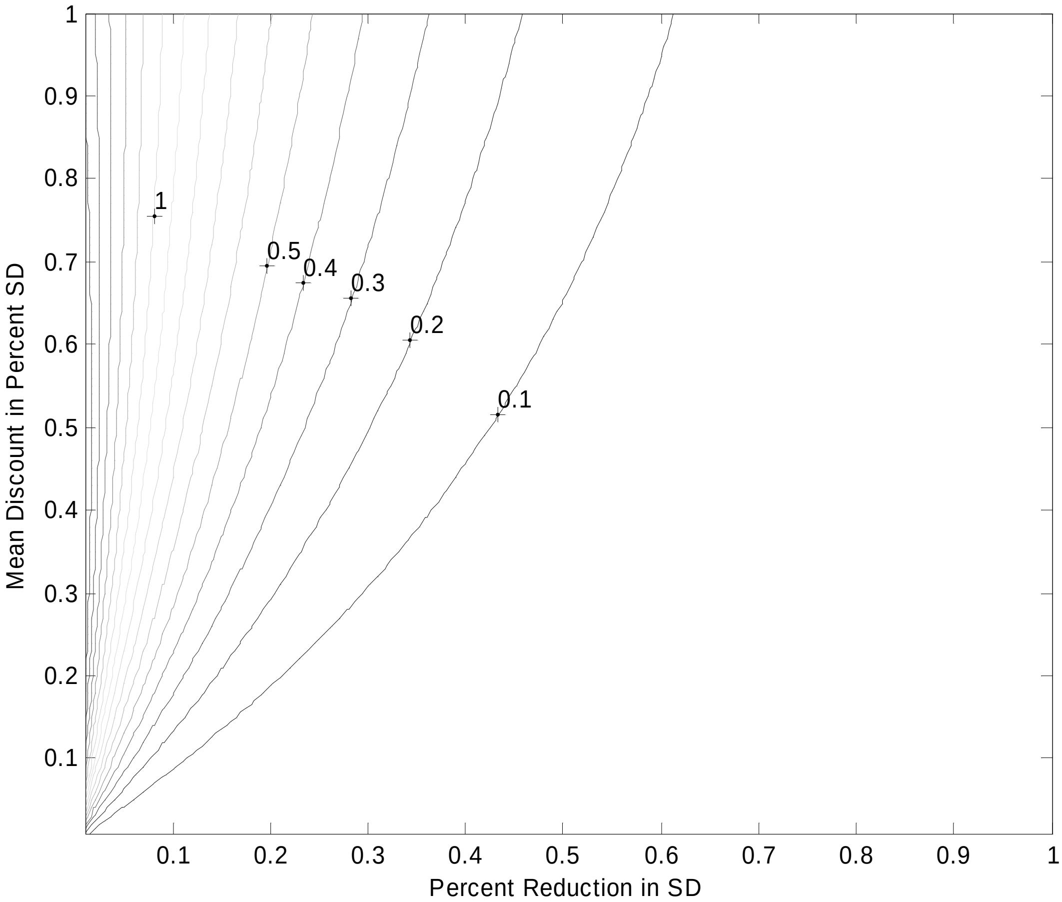
$\rho =$	0.05	0.10	0.15	0.20	0.25
$\gamma =$	0.667	0.429	0.250	0.111	0.000

Table 4. Values of $\tilde{\gamma}$ for various values of α, δ.[a]

Percent Reduction in SD $1-\alpha=$	Mean Discount in Percent SD $\delta 2\sqrt{3}=$				
	0.1	**0.2**	**0.3**	**0.4**	**0.5**
0.9	**0.77**	**0.60**	**0.46**	**0.35**	**0.26**
0.8	**0.75**	**0.56**	**0.41**	**0.31**	**0.22**
0.7	**0.72**	**0.51**	**0.36**	**0.25**	**0.17**
0.6	**0.68**	**0.46**	**0.31**	**0.20**	**0.12**
0.5	**0.63**	**0.39**	**0.24**	**0.14**	**0.07**
0.4	**0.56**	**0.31**	**0.16**	**0.07**	**0.03**
0.3	**0.46**	**0.20**	**0.07**	**0.02**	**0.00**
0.2	**0.31**	**0.07**	**0.01**	**-**	**-**
0.1	**0.07**	**-**	**-**	**-**	**-**
0.0	**-**	**-**	**-**	**-**	**-**

[a] The discount of δD_g is equal to $2\sqrt{3}\delta$ times the standard deviation of $g(x)$. Similarly, $1-\alpha$ measures the percentage reduction in standard deviation of $g(x)$. The mark “-“ indicates the maximum value is negative, or that the action cannot be rationalized with a monotonically increasing utility function.

Figure 1. The difference between $\tilde{\gamma}$ and γ.[a]



[a.] Value increases as one moves northwest with values $\tilde{\gamma} - \gamma = 0.1, 0.2, 0.3, \ldots$.