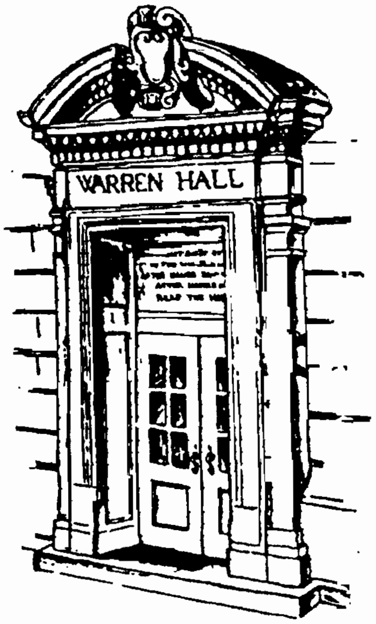

WP 94-14
December 1994

Working Paper

Department of Agricultural, Resource, and Managerial Economics
Cornell University, Ithaca, New York 14853-7801 USA

USING VARIATIONAL INEQUALITIES TO SOLVE SPATIAL PRICE EQUILIBRIUM MODELS WITH *AD VALOREM* TARIFFS AND ACTIVITY ANALYSIS

Charles F. Nicholson
Phillip M. Bishop
and
Anna Nagurney

PREFACE

Charles Nicholson and Phillip Bishop are graduate students in the Department of Agricultural, Resource and Managerial Economics at Cornell University. Anna Nagurney is a professor at the School of Management, University of Massachusetts, Amherst, Massachusetts 01003.

The authors gratefully acknowledge the helpful comments of Andrew Novakovic and James Pratt on earlier drafts of this paper, and the efforts of Wendy Barrett to prepare the final copy of the document.

The numerical examples presented in this paper were solved at the Cornell National Supercomputing Facility using GAMS 2.25. The authors wish to acknowledge support for this project from the The Dairy Program of the National Institute for Livestock and Dairy Policy (NILDP), GAMS Development Corporation, and the Cornell National Supercomputing Facility (CNSF), a resource of the Cornell Theory Center funded in part by the National Science Foundation, New York State, the IBM Corporation, and members of the Center's Corporate Research Institute.

Additional copies of this Working Paper can be requested from the publications clerk in the Department of Agricultural, Resource, and Managerial Econoimcs, 52 Warren Hall, Cornell University, Ithaca, NY 14853-7801.

TABLE OF CONTENTS

ABSTRACT

Spatial price equilibrium (SPE) models are commonly-used tools for the spatial analysis of commodity trade. The optimization methods often used to solve SPE problems cannot, in general, be applied in the presence of *ad valorem* tariffs. In this paper, we briefly introduce an alternative method for solving SPE problems with *ad valorem* tariffs, the method of variational inequalities (VI). We discuss the relationship between VI and optimization methods, showing how incorporation of *ad valorem* tariffs violates the conditions necessary for formulation of the SPE problem as an optimization problem. We demonstrate the VI formulation for two general SPE problems, one with explicit, price-responsive supply functions, and the other with product supplies determined from the efficient choice of production technologies (activity analysis). A modified projection algorithm proven to converge to an optimal solution results in computationally simple closed-form expressions, even for nonlinear SPE problems. To demonstrate the applicability of the VI method to the analysis of spatial commodity trade, we present three numerical examples of SPE problems. Our method of solving SPE models with *ad valorem* tariffs is applicable to modeling trade agreements such as NAFTA and GATT in which tariffication plays an important role.

Using Variational Inequalities to Solve Spatial Price Equilibrium Models with *Ad Valorem* Tariffs and Activity Analysis

Introduction

In the early 1950s, Samuelson (1952) demonstrated that market equilibria could be computed using optimization techniques. During the 1960s and 1970s, applied economists devoted much effort to formulating empirical market equilibrium models, and to the development and refinement of methods for their solution. By the late 1960s, spatial price equilibrium (SPE) models had become commonly-used tools for spatial analysis of commodity trade (cf. Judge and Takayama, 1973; Thompson, 1989), and optimization methods such as quadratic programming (QP) were most frequently used for their solution. The QP approach, however, was limited to problems characterized by linear and "symmetric" supply and demand price functions.

By the early 1970s, however, the limitations of optimization methods for solution of SPE problems were widely appreciated. In particular, Takayama and Judge (1971) noted that optimization methods could not be applied to solve SPE problems involving supply and demand functions with asymmetric cross-price terms, or SPE problems incorporating discriminatory *ad valorem* tariffs (differential tariff rates imposed by the importing country on exporting countries). Methods of computing solutions to SPE problems that optimization methods could not solve thus received greater attention during the 1970s and 1980s. These methods include fixed-point algorithms, complementarity programming, and variational inequality methods.

Over 20 years ago, Takayama and Judge (1971) employed linear complementarity techniques to solve a SPE problem with linear, asymmetric price functions and non-discriminatory *ad valorem* tariffs. At about the same time, nonlinear complementarity methods were under development (Cottle et al., 1970), but to date these methods have been proven convergent for a limited number of applications (cf. Matheisen, 1985). Holland and Sharples (1984) applied the Vector Sandwich Method of Kuhn and MacKinnon (1975) to solve a SPE model of world wheat trade including *ad valorem* tariffs. Preckel (1985) noted, however, that the fixed-point approach for solving large-scale equilibrium problems is often less computationally efficient than other methods.

Ginsburgh and Van der Heyden (1988) noted that an iterative optimization method could be applied to solve SPE models with discriminatory *ad valorem* tariffs. However, this method cannot be shown theoretically to converge to an optimal solution, although it appears to converge often in empirical applications. Most recently, Nagurney *et al.* (1995b) developed a variational inequality formulation for SPE problems with discriminatory *ad valorem* tariffs, proving global convergence for problems with certain characteristics, and exploring computational aspects of small and large SPE problems.

Since the 1960s, the conditions under which the various algorithms applied to solve SPE problems could be proven to converge to a set of equilibrium conditions has been a continuous theme in the literature. Methods such as reactive programming, which came into use for a brief period in the early 1960s, were supplanted by methods such as QP, which could be theoretically shown to converge to the equilibrium conditions under suitable linearity, symmetry, and convexity assumptions. Convergence conditions remain an important aspect of the choice of methodologies for the computation of solutions to SPE problems.

The importance of incorporating characteristics, such as asymmetric cross-price terms and discriminatory *ad valorem* tariffs, that limit the use of optimization methods for spatial economic models has been underscored by recent developments in trade policy. The emphasis on the conversion of non-tariff trade barriers, such as quotas, to tariffs as a mechanism to liberalize trade implies that spatial economic models of trade liberalization must permit explicit analysis of changes in *ad valorem* tariff rates. Both the North American Free Trade Agreement (NAFTA) and the recent General Agreement on Tariffs and Trade (GATT) negotiations, for example, specify tariffication as the primary mechanism to lower trade barriers.

This paper has three principal objectives: 1) to briefly introduce the method of variational inequalities (VI) and explain its relationship to optimization methods, 2) to demonstrate the VI formulation for two SPE problems, one with explicit, price-responsive product supply functions, and the other with product supplies determined from the efficient choice of production technologies (activity analysis), and 3) to illustrate the computational simplicity of the VI approach for three examples of SPE problems with discriminatory *ad valorem* tariffs. A secondary objective of this paper is to contribute to the awareness of issues (such as ease of implementation, computational efficiency, and convergence conditions) important to the selection of appropriate methods for the computation of solutions to SPE problems.

The Variational Inequality Problem

The variational inequality problem is a general problem formulation that encompasses many mathematical problems, including optimization problems, complementarity problems, and fixed-point problems (Nagurney, 1993). Variational inequalities were developed originally to study certain types of partial differential equations defined over infinite-dimensional spaces (Kinderlehrer and Stampacchia, 1980). For most economic equilibrium problems, a finite-dimensional variational inequality problem is appropriate and can be defined as follows:

Definition 1. The finite-dimensional variational inequality problem, $VI(F, X)$, is to determine a vector $x^* \in X \subset R^n$, such that

$$F(x^*)^T \cdot (x - x^*) \geq 0, \quad \forall\, x \in X, \tag{1}$$

where F is a given continuous function from X to R^n, X is a given closed, convex set, and ^{T} indicates transpose.

The book by Nagurney (1993) contains basic results concerning the existence and uniqueness of the solutions to the finite-dimensional VI problem, and discusses the sensitivity of solutions to changes in the parameters of $F(x)$.

The VI formulation is convenient because it allows unified treatment of equilibrium problems and optimization problems. When the function $F(x)$ in $VI(F, X)$ has certain properties, a direct relationship exists between $VI(F, X)$ and a specific optimization problem. Following Nagurney (1993), which contains the proofs, we offer the following propositions.

Proposition 1. If x^* is a solution to the problem: *Minimize f(x), subject to* $x \in X$, where $f(x)$ is continuously differentiable, and X is a closed, convex set, then x^* is a solution to the VI problem $VI(\nabla f, X)$

$$\nabla f(x^*)^T(x-x^*) \geq 0, \quad \forall\, x \in X. \tag{2}$$

An additional proposition relates the solution of $VI(\nabla f, X)$ to the solution of the minimization problem.

Proposition 2. If $f(x)$ is convex and x^* is a solution to $VI(\nabla f, X)$, then x^* is a solution to the problem: *Minimize f(x), subject to* $x \in X$.

When $F = \nabla f$, we say that F is a *gradient mapping*. When F is a gradient mapping, then the variational inequality problem $VI(F, X)$ is exactly equal to the conditions for a solution to the optimization problem. In this case, an economic equilibrium problem can be formulated either as an optimization problem or as a VI problem. When F is not a gradient mapping, this connection no longer exists between the VI problem and the optimization problem. Thus, if the VI formulation $VI(F, X)$ involves an F that is not a gradient mapping, no equivalent optimization problem can be formulated. It is important to note, hence, that the VI problem is the more general one.

The function F is a gradient mapping if it is *symmetric*; this terminology arises from the *symmetry principle*, which states that F is a gradient mapping if and only if its Jacobian matrix is symmetric. Equivalently, F is a gradient mapping if F is *integrable*. The following theorem contained in Harker (1993) gives two necessary and sufficient conditions for a continuously differentiable function to be a gradient mapping on an open domain R^n.

Theorem 1. Let $F: S \mapsto R^n$ be continuously differentiable on the open convex set $S \subseteq R^n$. Then the following three statements are equivalent:

i) there exists a real-valued function f such that $F(x) = \nabla f(x)$, $\forall\, x \in S$,

ii) the Jacobian matrix $\nabla F(x)$ is symmetric $\forall\, x \in S$,

iii) F is integrable in S.

The implication of Propositions 1 and 2 and Theorem 1 is that if the VI formulation of equilibrium conditions is characterized by a function with a symmetric Jacobian matrix, then the solution of the equilibrium conditions and the solution of a particular optimization problem are one and the same. When the variational inequality formulation involves a function whose Jacobian matrix $\nabla F(x)$ is not symmetric, then no equivalent optimization problem can be specified to solve the problem $VI(F, X)$.

The SPE Model with *Ad Valorem* Tariffs

In this section we introduce the spatial market model with *ad valorem* tariffs. We state the governing equilibrium conditions and then derive the variational inequality formulation. The model in the absence of tariffs simplifies to the well-known spatial price equilibrium models pioneered by Samuelson (1952) and Takayama and Judge (1971).

Consider I supply markets involved in the production of a homogeneous commodity and J demand markets. Denote a typical supply market by i and a typical demand market by j. Let s_i denote the supply at market i and d_j the demand at demand market j. Group the supplies into a column vector $s \in R^I$ and the demands into a column vector $d \in R^J$. Let Q_{ij} denote the nonnegative commodity shipment between supply and demand market pair (i,j), and group the commodity shipments into a column vector $Q \in R^{IJ}$. The commodity shipments and the supplies and demands must satisfy the following conservation of flow equations:

$$s_i = \sum_{j=1}^{J} Q_{ij}, \quad i=1, \dots , I, \tag{3}$$

or equivalently, in vector form,

$$s = AQ, \tag{4}$$

where A is an $(I \times IJ)$ node-arc incidence matrix with components $(A)_{ij} = 1$, for $j=1J,\dots,iJ$, and 0, otherwise, and

$$d_j = \sum_{i=1}^{I} Q_{ij}, \quad j=1, \dots, J, \tag{5}$$

or, equivalently,

$$d = BQ, \tag{6}$$

where B is the $(J \times IJ)$ matrix such that $B = (I_J \,|\, I_J \,|\, \dots \,|\, I_J)$ and where I_J denotes the $J \times J$ identity matrix. Thus, the quantity supplied at each supply market i must be equal to the sum of the commodity shipments from that market to all the demand markets j, and the quantity demanded at each demand market j must be equal to the sum of the commodity shipments from the supply markets i to each demand market.

We now describe the price and cost structure. Let π_i denote the supply price at supply market i and ρ_j the demand price at demand market j. Group the supply prices into a row vector $\pi \in R^I$ and the demand prices into a row vector $\rho \in R^J$. The transportation cost associated with shipping the commodity from supply market i to demand market j is denoted by c_{ij}. Group the transportation costs into a row vector $c \in R^{IJ}$.

The supply price at a supply market may, in general, depend upon the supplies of the commodity at every supply market, that is,

$$\pi = \pi(s). \tag{7}$$

Similarly, the demand price at a demand market may, in general, depend on the demands for the commodity at every demand market, that is,

$$\rho = \rho(d). \tag{8}$$

Note that these static supply and demand price functions assume no uncertainty and ignore seasonal variation in production and consumption.

The per-unit transportation cost associated with shipping the commodity between a pair of supply and demand markets is assumed to be fixed, that is, it is independent of the volume of commodity shipments, although specification of per-unit costs as a function of Q is a straightforward extension of the model presented here. The fixed generalized per-unit transfer cost is denoted by $\bar{c}_{ij}$, and the associated IJ-dimensional vector by c. Thus,

$$c = \bar{c} \ . \tag{9}$$

Note that other fixed per-unit transfer costs and per-unit tariffs (or other per-unit taxes such as fixed export taxes) can be readily incorporated into the fixed $\bar{c}$ function.

In the absence of policy interventions and under the assumption of perfect competition, the well-known spatial-price equilibrium conditions (cf. Samuelson (1952), Takayama and Judge (1971)) are as follows: for all pairs of supply and demand markets (i,j); $i=1,\ldots,I$; $j=1,\ldots,J$, a commodity supply, shipment, and demand pattern (s^*,Q^*,d^*) satisfying constraints (3) and (5) is in equilibrium if

$$\pi_i(s^*) + \bar{c}_{ij} \begin{cases} = \rho_j(d^*), & \textit{if} \quad Q^*_{ij}>0 \\ \geq \rho_j(d^*), & \textit{if} \quad Q^*_{ij}=0 \ . \end{cases} \tag{10}$$

Thus, in equilibrium, if a strictly positive amount of the commodity is shipped between a pair of supply and demand markets, then the supply price at the supply market plus the cost of transportation must equal the demand price at the demand market. There will be no shipment of the commodity between a pair of markets if the sum of the supply price and transfer costs exceeds the demand price.

As established in Florian and Los (1982), who considered a general network and nonfixed transportation costs, the above equilibrium conditions can be formulated as a variational inequality problem, as stated in the following theorem.

Theorem 2. A pattern (s^*, Q^*, d^*) satisfying (10), subject to constraints (3) and (5) is a spatial price equilibrium if and only if it satisfies the variational inequality

$$\pi(s^*) \cdot (s - s^*) + \bar{c} \cdot (Q - Q^*) - \rho(d^*) \cdot (d - d^*) \geq 0, \tag{11}$$

for all (s, Q, d) satisfying (3) and (5).

We digress for a moment to note the form of the Jacobian matrix of $F(s,Q,d)$ for this variational inequality, which is given by

$$\nabla F(s, Q, d) = \begin{bmatrix} \frac{\partial \pi_1}{\partial s_1} & \cdots & \frac{\partial \pi_1}{\partial s_I} & \frac{\partial \pi_1}{\partial Q_{11}} & \cdots & \frac{\partial \pi_1}{\partial Q_{IJ}} & \frac{\partial \pi_1}{\partial d_1} & \cdots & \frac{\partial \pi_1}{\partial d_J} \\ \vdots & \ddots & & & & & & & \vdots \\ \frac{\partial \pi_I}{\partial s_1} & \cdots & & & & & & \cdots & \frac{\partial \pi_I}{\partial d_J} \\ \frac{\partial c_{11}}{\partial s_1} & \cdots & & & & & & \cdots & \frac{\partial c_{11}}{\partial d_J} \\ \vdots & & & & & & & & \vdots \\ \frac{\partial c_{IJ}}{\partial s_1} & \cdots & & & & & & \cdots & \frac{\partial c_{IJ}}{\partial d_J} \\ -\frac{\partial \rho_1}{\partial s_1} & \cdots & & & & & & \cdots & -\frac{\partial \rho_1}{\partial d_J} \\ \vdots & & & & & & & \ddots & \vdots \\ -\frac{\partial \rho_J}{\partial s_1} & \cdots & -\frac{\partial \rho_J}{\partial s_I} & -\frac{\partial \rho_J}{\partial Q_{11}} & \cdots & -\frac{\partial \rho_J}{\partial Q_{IJ}} & -\frac{\partial \rho_J}{\partial d_1} & \cdots & -\frac{\partial \rho_J}{\partial d_J} \end{bmatrix} \tag{12}$$

where we typically assume that

$$\frac{\partial \pi_i}{\partial Q_{ij}} = \frac{\partial \pi_i}{\partial d_j} = 0; \ \frac{\partial c_{ij}}{\partial s_i} = \frac{\partial c_{ij}}{\partial d_j} = 0; \ \frac{\partial \rho_j}{\partial s_i} = \frac{\partial \rho_j}{\partial Q_{ij}} = 0. \tag{13}$$

The partials of Q_{ij} with respect to c_{ij} are all zero because the $c_{ij} = \bar{c}$ are fixed for all (i,j). Hence, the Jacobian matrix has the block form

$$\nabla F(s, Q, d) = \begin{bmatrix} \frac{\partial \pi}{\partial s} & 0 & 0 \\ 0 & \frac{\partial c}{\partial Q} = 0 & 0 \\ 0 & 0 & -\frac{\partial \rho}{\partial d} \end{bmatrix}. \tag{12a}$$

If one makes the following assumption:

$$\frac{\partial \pi_i}{\partial s_k} = \frac{\partial \pi_k}{\partial s_i}, \ \forall \ i \neq k; \ \frac{\partial \rho_j}{\partial d_l} = \frac{\partial \rho_l}{\partial d_j}, \ \forall \ j \neq l, \tag{14}$$

by Theorem 1, this (special) problem can be formulated as an optimization problem, as Samuelson (1952) and Takayama and Judge (1971) have long since noted.

We now introduce discriminatory *ad valorem* tariffs into the above model. Let τ_{ij} denote the *ad valorem* tariff, assumed nonnegative, and applied by demand market j to imports from supply market i. The incorporation of *ad valorem* tariffs modifies the spatial price equilibrium conditions as follows: For all pairs of supply and demand markets (i,j); $i=1,\ldots,I$; $j=1,\ldots,J$, a commodity supply, shipment, and demand pattern (s^*,Q^*,d^*) satisfying (3) and (5) is said to be in equilibrium if

$$(\pi_i(s^*) + \bar{c}_{ij})\cdot(1+\tau_{ij}) \begin{cases} = \rho_j(d^*), & \text{if } Q^*_{ij}>0 \\ \geq \rho_j(d^*), & \text{if } Q^*_{ij}=0 . \end{cases} \tag{15}$$

Thus, in equilibrium, if a strictly positive amount of the commodity is shipped between a pair of supply and demand markets, then the effective supply price plus transportation cost after the imposition of the *ad valorem* tariff must be equal to the demand price at the demand market. If there is no commodity shipment between a pair of supply and demand markets, then the effective supply price plus transfer cost can exceed the demand price.

Given constraints (4) and (6), we can define the functions

$$\begin{aligned} \hat{\pi}_i &\equiv \pi_i(AQ), \quad i = 1, \ldots, I, \\ \hat{\rho}_j &\equiv \rho_j(BQ), \quad j = 1, \ldots, J. \end{aligned} \tag{16}$$

The variational inequality formulation of the equilibrium conditions governing the SPE problem with *ad valorem* tariffs can be stated in the following theorem, the proof of which is included for easy reference.

Theorem 3 (Nagurney, Nicholson, and Bishop, 1995b).
A commodity shipment pattern $Q^* \in R^{mn}_+$ is an equilibrium pattern, that is, satisfies conditions (15) if and only if it satisfies the variational inequality problem

$$\sum_{i=1}^{I}\sum_{j=1}^{J} ((\hat{\pi}_i(Q^*)+\bar{c}_{ij})\cdot(1+\tau_{ij}) - \hat{\rho}_j(Q^*))\cdot(Q_{ij} - Q^*_{ij}) \geq 0, \quad \forall Q \in R^{IJ}_+. \tag{17}$$

Proof:

We first establish that if a pattern Q^* satisfies equilibrium conditions (15), then it also satisfies variational inequality (17). For a fixed market pair *(i,j)*, (15) implies that

$$((\pi_i(s^*) + \bar{c}_{ij})\cdot(1 + \tau_{ij}) - \rho_j(d^*))\cdot(Q_{ij} - Q_{ij}^*) \geq 0, \; \forall \; Q_{ij} \geq 0, \tag{18}$$

or equivalently,

$$((\hat{\pi}_i(Q^*) + \bar{c}_{ij})\cdot(1 + \tau_{ij}) - \hat{\rho}_j(Q^*))\cdot(Q_{ij} - Q_{ij}^*) \geq 0, \quad \forall \; Q_{ij} \geq 0. \tag{19}$$

Summing over all market pairs *(i,j)*, we obtain the variational inequality (17).

We now show that any solution to the variational inequality (17) also satisfies the equilibrium conditions (15). In variational inequality (17), set $Q_{ij}=Q^*_{ij}$, for all market pairs $(i,j) \neq (k,l)$. Then, (17) reduces to

$$((\hat{\pi}_k(Q^*) + \bar{c}_{kl})\cdot(1 + \tau_{kl}) - \hat{\rho}_l(Q^*))\cdot(Q_{kl} - Q_{kl}^*) \geq 0, \;\; \forall \; Q_{kl} \geq 0, \tag{20}$$

or,

$$((\pi_k(s^*) + \bar{c}_{kl})\cdot(1 + \tau_{kl}) - \rho_l(d^*))\cdot(Q_{kl} - Q_{kl}^*) \geq 0, \;\; \forall \; Q_{kl} \geq 0, \tag{21}$$

which implies equilibrium conditions (15) for market pair *(k,l)*. Because this statement is independent of how we select the pair, the conditions hold for all market pairs. The proof is complete.

We now note the form of the Jacobian matrix for *F(Q)*. Because

$$F_{ij}(Q) \equiv (\hat{\pi}_i(Q) + \bar{c}_{ij})(1+\tau_{ij}) - \hat{\rho}_j(Q), \tag{22}$$

the elements of the Jacobian matrix $\nabla F(Q)$ are

$$\frac{\partial F_{ij}(Q)}{\partial Q_{ij}} = \frac{\partial \hat{\pi}_i(Q)}{\partial Q_{ij}}(1 + \tau_{ij}) - \frac{\partial \hat{\rho}_j(Q)}{\partial Q_{ij}}, \tag{23}$$

and the Jacobian matrix is

$$\nabla F(Q) = \begin{bmatrix} \frac{\partial \hat{\pi}_1}{\partial Q_{11}}(1+\tau_{11}) - \frac{\partial \hat{\rho}_1}{\partial Q_{11}} & \frac{\partial \hat{\pi}_1}{\partial Q_{12}}(1+\tau_{11}) - \frac{\partial \hat{\rho}_1}{\partial Q_{12}} & \cdots & \frac{\partial \hat{\pi}_1}{\partial Q_{IJ}}(1+\tau_{11}) - \frac{\partial \hat{\rho}_1}{\partial Q_{IJ}} \\ \frac{\partial \hat{\pi}_1}{\partial Q_{11}}(1+\tau_{12}) - \frac{\partial \hat{\rho}_2}{\partial Q_{11}} & \frac{\partial \hat{\pi}_1}{\partial Q_{12}}(1+\tau_{12}) - \frac{\partial \hat{\rho}_2}{\partial Q_{12}} & \cdots & \vdots \\ \vdots & & \ddots & \\ \frac{\partial \hat{\pi}_I}{\partial Q_{11}}(1+\tau_{IJ}) - \frac{\partial \hat{\rho}_J}{\partial Q_{11}} & \cdots & & \frac{\partial \hat{\pi}_I}{\partial Q_{IJ}}(1+\tau_{IJ}) - \frac{\partial \hat{\rho}_J}{\partial Q_{IJ}} \end{bmatrix}. \tag{24}$$

If the cross-price terms are symmetric (or zero), then

$$\frac{\partial \hat{\pi}_i}{\partial Q_{ij}} = \frac{\partial \hat{\pi}_i}{\partial Q_{ij'}}, \quad \frac{\partial \hat{\pi}_i}{\partial Q_{i'j'}} = \frac{\partial \hat{\pi}_{i'}}{\partial Q_{ij}}, \quad \frac{\partial \hat{\rho}_j}{\partial Q_{ij}} = \frac{\partial \hat{\rho}_j}{\partial Q_{i'j}}, \quad \frac{\partial \hat{\rho}_j}{\partial Q_{i'j'}} = \frac{\partial \hat{\rho}_{j'}}{\partial Q_{ij}} \tag{25}$$

for all $i \neq i'$ and $j \neq j'$, and the Jacobian matrix would be symmetric in the absence of *ad valorem* tariffs. With *ad valorem* tariffs, the Jacobian matrix is not symmetric unless the tariff rates are equal for all (i,j) pairs. Thus, in general, the SPE problem with *ad valorem* tariffs cannot be formulated as an optimization problem[1]. Note that asymmetry of the Jacobian matrix can also arise from asymmetric cross-price terms in $\pi(s)$ and $\rho(d)$, and it again follows that no optimization problem can be specified.

---

[1] When the same tariff rate is imposed by the importing country on all exporters, and this is true for all importers, an optimization problem based on excess supply and demand functions can be specified. For an example, see Cramer *et al.* (1991).

Variational inequality (17) is put into standard form (1) using (22). Let $F(Q) \in R^{IJ}$ be the row vector with the (i,j)-th component, $F_{ij}(Q)$. Then the variational inequality (17) can be expressed in standard form as: Determine $Q^* \in X$, such that

$$F(Q^*) \cdot (Q - Q^*) \geq 0, \quad \forall\, Q \in X, \tag{26}$$

where the feasible set $X \equiv \{Q \mid Q \in R^{IJ}_+\}$.

We now discuss certain qualitative properties of the function $F(Q)$ that are useful in establishing convergence of the computational procedure for certain supply and demand functions, which can be both asymmetric and nonlinear. In particular, we give conditions that guarantee monotonicity and Lipschitz continuity of F.

Lemma 1 (Nagurney, Nicholson, and Bishop, 1995b). Assume that $\pi(s)$ and $-\rho(d)$ are strongly monotone in s and d, respectively, i.e.,

$$\begin{aligned} [\pi(s^1) - \pi(s^2)] \cdot [s^1 - s^2] &\geq \mu \| s^1 - s^2 \|^2, \quad \forall\, s^1, s^2 \in X, \\ -[\rho(d^1) - \rho(d^2)] \cdot [d^1 - d^2] &\geq \lambda \| d^1 - d^2 \|^2, \quad \forall\, d^1, d^2 \in X, \end{aligned} \tag{27}$$

for some $\mu, \lambda > 0$, then $F(Q)$ with components $F_{ij}(Q)$ defined in (22) is monotone, that is,

$$[F(Q^1) - F(Q^2)] \cdot [Q^1 - Q^2] \geq 0, \quad \forall\, Q^1, Q^2 \in X. \tag{28}$$

Note that the strong monotonicity condition holds in the simple case of linear, separable supply and demand price functions where

$$\begin{aligned} \pi_i(s_i) &= \alpha_{0i} + \alpha_{1i} \cdot s_i, \quad i = 1, \ldots, I, \\ \rho_j(d_j) &= \beta_{0j} - \beta_{1j} \cdot d_j, \quad j = 1, \ldots, J, \end{aligned} \tag{29}$$

with

$$\alpha_{1i}, \beta_{1j} > 0, \quad \forall\, i, j.$$

Lemma 2 (Nagurney, Nicholson, and Bishop, 1995b). *F(Q)* with components $F_{ij}(Q)$ in (22) is Lipschitz continuous on X, that is, there exists an $L > 0$, such that

$$\|F(Q^1) - F(Q^2)\| \leq L\|Q^1 - Q^2\|, \quad \forall\, Q^1, Q^2 \in X, \tag{30}$$

under the assumption that $F_{ij}(Q)$ has bounded first-order derivatives for all $Q \in X$. Further, they show that

$$\|\nabla F_{ij}(\tilde{Q}^{ij})\| \leq L_{ij}, \quad \forall\, i, j, \;\; \forall\, Q \in X. \tag{31}$$

where

$$\|\nabla F_{ij}(\tilde{Q}^{ij})\| \tag{32}$$

is the norm of the row vector of ∇ *F(Q)* (the Jacobian matrix of F) consisting of

$$\left[\frac{\partial F_{ij}(\tilde{Q}^{ij})}{\partial Q_{11}}, \; \dots \;, \; \frac{\partial F_{ij}(\tilde{Q}^{ij})}{\partial Q_{IJ}}\right] \tag{33}$$

or

$$\|\nabla F_{ij}(\tilde{Q}^{ij})\| = \left[\frac{\partial F_{ij}(\tilde{Q}^{ij})}{\partial Q_{11}}\right]^2 + \dots + \left[\frac{\partial F_{ij}(\tilde{Q}^{ij})}{\partial Q_{IJ}}\right]^2, \tag{34}$$

where

$$(\tilde{Q}^{ij}) = \theta_{ij}(Q^1) + (1 - \theta_{ij})(Q^2), \quad \forall\, 0<\theta_{ij}<1, \tag{35}$$

is a convex combination of Q^1 and Q^2, and the existence of θ_{ij} is guaranteed by the Mean Value Theorem. Thus, L can be determined as

$$L = \max_{\{i,j\}}\{ L_{ij} \} . \tag{36}$$

It follows that $F(Q)$ will have bounded first-order derivatives if the supply and demand price functions have bounded first-order derivatives and the tariff rates are finite.

An SPE Model Incorporating Activity Analysis

It often is useful to incorporate activity analysis into a SPE problem when a raw material undergoes transformation before being traded, when analysis of different production technologies in a trade context is of interest, or when insufficient information is available to specify continuous supply price functions for the supply markets. Takayama and Judge (1971) discuss a number of variations of SPE models using activity analysis. We now develop a VI formulation of a SPE problem incorporating both *ad valorem* tariffs and activity analysis. The model is analogous in many ways to the VI formulation of the SPE model with continuous supply price functions.

Consider I supply markets involved in the production of a single homogeneous commodity, J demand markets, K resources available to produce the commodity, and L possible production activities. The set of production activities may be thought of as the aggregate technology available to produce the commodity. Denote a typical supply market by i, a typical demand market by j, a typical resource by k, and a typical production activity by l. Let s_{lij} denote the amount of the good produced in region i by production activity l that is shipped to region j, d_j denote the demand at market j, and x_{ki} denote the amount of resource k used in region i. Let Q_{ij} denote the nonnegative commodity shipment between supply region i and demand region j. Group the input supplies into a vector $x \in R^{KI}$, the production activities into a vector $s \in R^{LIJ}$, the demands into a vector $d \in R^{J}$, and the commodity shipments into a vector $Q \in R^{IJ}$. Let $a_{klij} \geq 0$ denote the amount of input k required to produce one unit of s_{lij}, noting that this allows goods produced for different markets j to have different input compositions.

The resource use, commodity shipments, and the demands must satisfy the following conservation of flow equations:

$$
\begin{aligned}
x_{ki} &= \sum_{l=1}^{L}\sum_{j=1}^{J} a_{klij} \cdot s_{lij}, \quad k=1, \dots, K;\ i=1, \dots, I, \\
Q_{ij} &= \sum_{l=1}^{L} s_{lij}, \quad i=1, \dots, I;\ j = 1, \dots, J, \\
d_j &= \sum_{i=1}^{I} Q_{ij} = \sum_{l=1}^{L}\sum_{i=1}^{I} s_{lij}, \quad j=1, \dots, J.
\end{aligned}
\tag{37}
$$

or, equivalently, in vector form:

$$
\begin{aligned}
x &= As, \\
Q &= Bs, \\
d &= Cs,
\end{aligned}
\tag{38}
$$

for appropriately defined matrices A, B, and C. These constraints imply that the total use of input k in supply market i, x_{ki}, equals the sum across all production activities l and demand markets j of the requirements of that input per unit of production multiplied by the number of units produced. Also, shipments from supply market i to demand market j, the Q_{ij}, must equal the sum across production activities l, that is, the number of units produced in i for shipment to j. The sum across supply markets of commodity shipments to j must equal the quantity demanded at j.

To define the cost and price structures, let ω_{ki} denote the supply price of input x_{ki} in market i, and ρ_j the demand price for the final good in demand market j. The transportation cost associated with shipping the good from market i to market j is denoted c_{ij}. We assume that the supply price of input x_{ki} may, in general, depend upon the demands for the input at every supply market, that is,

$$\omega = \omega(x). \tag{39}$$

Similarly, the demand price at a demand market may, in general, depend upon the demands for the final commodity at every demand market, that is,

$$\rho = \rho(d). \tag{40}$$

These static input supply and product demand price functions assume no uncertainty and ignore seasonal variation in production and consumption.

The per-unit cost of transportation associated with shipping the commodity between a pair of supply and demand markets can be assumed to be a function of the total volume of commodity shipped between all markets (i,j), or

$$c_{ij} = c_{ij}(Q), \quad i = 1, ..., I; \; j = 1, ..., J, \tag{41}$$

where c is a IJ-dimensional vector. Note that other fixed per-unit transfer costs and per-unit tariffs can be readily incorporated into the $c_{ij}(Q)$ function.

In the absence of policy interventions, and under the assumption of perfect competition in input and final goods markets, the spatial price equilibrium conditions for this problem can be stated as follows: For all combinations of supply markets, production activities, and demand markets (l,i,j); $l=1,\ldots,L$; $i=1,\ldots,I$; $j=1,\ldots,J$, an input use, production, commodity shipment, and demand pattern (x^*, s^*, Q^*, d^*) satisfying the constraints in (37) is in equilibrium if

$$\sum_k a_{klij} \cdot \omega_{ik}(x^*) + c_{ij}(Q^*) \begin{cases} = \rho_j(d^*), \text{ if } s^*_{lij} > 0, \\ \geq \rho_j(d^*), \text{ if } s^*_{lij} = 0. \end{cases} \tag{42}$$

Given constraints (38), we can define the functions

$$\begin{aligned} \hat{\omega}_{ki} &\equiv \omega_{ki}(As), \quad k=1,\dots,K;\ i=1,\dots,I, \\ \hat{c}_{ij} &\equiv c_{ij}(Bs), \quad i=1,\dots,I;\ j=1,\dots,J, \\ \hat{\rho}_j &\equiv \rho_j(Cs), \quad j=1,\dots,J. \end{aligned} \tag{43}$$

Thus, the equilibrium conditions (42) can be expressed as

$$\sum_k a_{klij} \cdot \hat{\omega}_{ki}(s^*) + \hat{c}_{ij}(s^*) \begin{cases} = \hat{\rho}_j(s^*), \text{ if } s^*_{lij} > 0, \\ \geq \hat{\rho}_j(s^*), \text{ if } s^*_{lij} = 0, \end{cases} \tag{44}$$

with appropriate substitutions from (43).

Thus, in equilibrium, if a positive amount of the commodity is produced by activity l and shipped from supply market i to demand market j, the total value of the inputs used in production plus the cost of transportation from i to j is equal to the product demand price in region j. If the value of the inputs used in production plus the transportation cost from i to j exceed the product demand price in the demand market, no commodity is shipped from supply market i to demand market j.

We are now ready to derive the variational inequality formulation of the equilibrium conditions governing the spatial price equilibrium model incorporating activity analysis. We state the following:

Theorem 4. A commodity production pattern $s^* \in R^{LIJ}$ is an equilibrium pattern, that is it satisfies conditions (44) if and only if it satisfies the variational inequality problem

$$\sum_l \sum_i \sum_j [\sum_k a_{klij} \cdot \hat{\omega}_{ki}(s^*) + \hat{c}_{ij}(s^*) - \hat{\rho}_j(s^*)] \cdot (s_{lij} - s^*_{lij}) \geq 0, \quad \forall s \in R^{LIJ}_+. \tag{45}$$

Proof:

We first establish that if a pattern s^* satisfies equilibrium conditions (44) then it also satisfies variational inequality (45). For a fixed (l,i,j), (44) implies that

$$[\sum_k a_{klij} \cdot \hat{\omega}_{ki}(s^*) + \hat{c}_{ij}(s^*) - \hat{\rho}_j(s^*)](s_{lij} - s^*_{lij}) \geq 0, \tag{46}$$

because if $s^*_{lij} > 0$, then the term in brackets, $[\bullet]$, equals zero and (46) is satisfied. If $s^*_{lij} = 0$, then $[\bullet]$ is greater than or equal to zero, and is multiplied by s_{lij}, which is greater than or equal to zero, so again the condition (46) is satisfied. If (46) holds for any (l,i,j), the summation across all (l,i,j) also holds, and this summation equals the variational inequality (45).

Now we show that any solution to variational inequality (45) also satisfies equilibrium condition (44). In (45), set $s_{lij} = s^*_{lij}$ for all $(l,i,j) \neq (l',i'j')$. Then (45) reduces to

$$[\sum_k a_{kl'i'j'} \cdot \hat{\omega}_{ki'}(s^*) + \hat{c}_{i'j'}(s^*) - \hat{\rho}_{j'}(s^*)] \cdot (s_{l'i'j'} - s^*_{l'i'j'}) \geq 0. \tag{47}$$

which implies equilibrium conditions (44) for *(l′,i′,j′)*. Because this statement is independent of how we select the *(l,i,j)*, the condition holds for all *(l,i,j)*. The proof is complete.

The variational inequalities formulation with *ad valorem* tariffs is given by a minor modification to (45), or

$$\sum_l \sum_i \sum_j [(\sum_k a_{klij} \cdot \hat{\omega}_{ki}(s^*) + \hat{c}_{ij}(s^*)) \cdot (1 + \tau_{ij}) - \hat{\rho}_j(s^*)]] \cdot (s_{lij} - s^*_{lij}) \geq 0, \quad \forall s \in R_+^{IJJ}. \tag{48}$$

where the τ_{ij} are the *ad valorem* tariff rates applied by demand market j to imports from supply market i.

The qualitative properties of $F(x)$ in this case can be derived in a manner analogous to that used for the *VI(F, X)* formulation of the SPE with continuous product supply price functions. Define

$$F_{lij}(s) \equiv (\sum_{k=1}^{K} a_{klij} \cdot \hat{\omega}_{ki}(s^*) + \hat{c}_{ij}(s^*)) \cdot (1 + \tau_{ij}) - \hat{\rho}_j(s^*) \quad \forall\ l,i,j, \tag{49}$$

and let $F(s) \in R^{IJJ}$ be the row vector with *(l,i,j)*-th component $F_{lij}(s)$. We now demonstrate that $F(s)$ with components $F_{lij}(s)$ is monotone and Lipschitz continuous on X, under suitable assumptions.

Lemma 3. Assume that $\omega(x)$, $c(Q)$, and $\rho(d)$ are each strongly monotone in x, Q, and d, respectively, that is,

$$\begin{aligned}
[\omega(x^1) - \omega(x^2)] \cdot [x^1 - x^2] &\geq \mu \| x^1 - x^2 \|^2, \quad \forall\ x^1, x^2, \\
[c(Q^1) - c(Q^2)] \cdot [Q^1 - Q^2] &\geq \lambda \| Q^1 - Q^2 \|^2, \quad \forall\ Q^1, Q^2, \\
-[\rho(d^1) - \rho(d^2)] \cdot [d^1 - d^2] &\geq \delta \| d^1 - d^2 \|^2, \quad \forall\ d^1, d^2,
\end{aligned} \tag{50}$$

for some μ, λ, and $\delta > 0$. Then $F(s)$ with components $F_{lij}(s)$ defined in (49) is monotone, i.e.,

$$[F(s^1) - F(s^2)]\cdot[(s^1) - (s^2)] \geq 0, \quad \forall\, s^1, s^2 \in X. \tag{51}$$

Proof:

We use the fact that if $\nabla F(s)$ is positive semidefinite for all $s \in X$, then $F(s)$ is monotone. The Jacobian matrix of $F(s)$ is given by

$$\nabla F(s) = \begin{bmatrix} \frac{\partial F_{111}}{\partial s_{111}} & \cdots & \frac{\partial F_{111}}{s_{LIJ}} \\ \vdots & & \vdots \\ \frac{\partial F_{LIJ}}{\partial s_{111}} & \cdots & \frac{\partial F_{LIJ}}{\partial s_{LIJ}} \end{bmatrix} \tag{52}$$

or, equivalently, in terms of the component functions

$$\nabla F(s) = \begin{bmatrix} (a_{1111}\cdot\frac{\partial\hat{\omega}_{11}}{\partial s_{111}} + \cdots + a_{K111}\cdot\frac{\partial\hat{\omega}_{KI}}{\partial s_{111}} + \frac{\partial\hat{c}_{11}}{\partial s_{111}})(1+\tau_{11}) - \frac{\partial\hat{\rho}_1}{\partial s_{111}} & \cdots \\ \vdots & \\ (a_{1LIJ}\cdot\frac{\partial\hat{\omega}_{1I}}{\partial s_{111}} + \cdots + a_{KLIJ}\cdot\frac{\partial\hat{\omega}_{KI}}{\partial s_{111}} + \frac{\partial\hat{c}_{IJ}}{\partial s_{111}})(1+\tau_{IJ}) - \frac{\partial\hat{\rho}_J}{\partial s_{111}} & \cdots \end{bmatrix} \tag{53}$$

$$= A^T\left[\frac{\partial\omega}{\partial x}\right]A + A^T\left[\frac{\partial\omega}{\partial x}\right]A\tau + B^T\left[\frac{\partial c}{\partial Q}\right]B + B^T\left[\frac{\partial c}{\partial Q}\right]B\tau + C^T\left[-\frac{\partial\rho}{\partial d}\right]C,$$

where A is a $KI \times LIJ$ matrix

$$A = \begin{bmatrix} A_{kl1j} & 0 & \cdots & 0 \\ 0 & A_{kl2j} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{klIj} \end{bmatrix} \tag{54}$$

and the (A_{klij}) submatrices are given by

$$A_{klij} = \begin{bmatrix} a_{11i1} & \cdots & a_{11iJ} & a_{12i1} & \cdots & a_{1LiJ} \\ a_{21i1} & \cdots & a_{21iJ} & a_{22i1} & \cdots & a_{2LiJ} \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ a_{K1i1} & \cdots & a_{k1iJ} & a_{k2i1} & \cdots & a_{KLiJ} \end{bmatrix} \tag{55}$$

and τ is the $(IJ \times IJ)$ dimensional matrix with diagonal components $\{\tau_{11}, \ldots, \tau_{1J}, \ldots, \tau_{I1}, \ldots, \tau_{IJ}\}$.

Under the assumption of strong monotonicity of ω, c, and $-\rho$,

$$\left[\frac{\partial \omega}{\partial x}\right], \quad \left[\frac{\partial c}{\partial Q}\right], \quad \text{and} \quad \left[-\frac{\partial \rho}{\partial d}\right]$$

are positive definite, and τ is a positive definite matrix. Therefore,

$$A^T\left[\frac{\partial \omega}{\partial x}\right]A, \quad A^T\left[\frac{\partial \omega}{\partial x}\right]A\tau, \quad B^T\left[\frac{\partial c}{\partial Q}\right]B, \quad B^T\left[\frac{\partial c}{\partial Q}\right]B\tau, \quad \text{and} \quad C^T\left[-\frac{\partial \rho}{\partial d}\right]C$$

are all positive semidefinite matrices. Moreover, because the sum of positive semidefinite matrices is a positive semidefinite matrix, it follows that $\nabla F(s)$ is positive semidefinite, and consequently, $F(s)$ is monotone. The proof is complete.

Lemma 4. $F(s)$ defined in (49) is Lipschitz continuous on X, that is, there exists an $L>0$ such that

$$\| F(s^1) - F(s^2) \| \leq L \| (s^1) - (s^2) \|, \quad \forall\, s^1, s^2 \in X \tag{56}$$

under the assumption that $F_{iij}(s)$ has bounded first-order derivatives for all $s \in X$. The proof is analogous to that for Lemma 2.

The Computational Procedure

We now discuss the computational procedure used to solve the SPE problem with discriminatory *ad valorem* tariffs described above when formulated as a variational inequality. The same algorithm will also be applied to the SPE problem with activity analysis. A general approach to solving the variational inequality $VI(F, X)$ consists of creating a sequence $\{x^k\} \subseteq X$ such that each $\{x^{k+1}\}$ solves an approximate and computationally simpler problem $VI(F, X^k)$ at each iteration $k=1,2,...$

$$F^k(x^{k+1})^T(x - x^{k+1}) \geq 0, \ \forall\, x \in X, \tag{57}$$

where $F^k(x)$ is some approximation to $F(x)$.

For a linear approximation,

$$F^k(x) = F(x^k) + A(x^k)(x - x^k), \tag{58}$$

where $A(x^k)$ is a $n \times n$ matrix. Several methods exist that differ in their choice of $A(x^k)$. For example, when $A(x^k) = \nabla F(x^k)$, this is called *Newton's method.* When a fixed, symmetric, positive definite matrix G is used, this is called the *projection method.* Thus, the *linear approximation method* consists of iteratively solving (57) for a suitable choice of $A(x^k)$. See Dafermos (1983) for a further discussion of the linear approximation method.

In what follows, we propose the modified projection method (MPM) of Korpelevich (1977) as the computational procedure for the solution of variational inequality (17) governing the spatial price equilibrium model with discriminatory *ad valorem* tariffs. Projection methods are linear approximation methods, and the MPM has less stringent conditions for convergence than the projection method. The MPM resolves the VI problem considered here into computationally simple sub-problems. In particular, one obtains a closed-form expression for the determination of commodity shipments at each iteration. The statement of the algorithm is as follows.

The Modified Projection Method

Step 0: Initialization. Start with $x^0 \in X$. Set the iteration number $k=1$ and select γ such that $0<\gamma<(1/L)$ where L is the Lipschitz constant defined in (30) and (36).

Step 1: Construction and Computation. Compute $\bar{x}^{k-1}$ by solving the variational inequality sub-problem:

$$[\bar{x}^{k-1} + (\gamma F(x^{k-1})^T - x^{k-1})]^T \cdot [x - \bar{x}^{k-1}] \geq 0, \quad \forall x \in X. \tag{59}$$

Step 2: Adaptation. Compute x^k by solving the variational inequality sub-problem:

$$[x^k + (\gamma F(\bar{x}^{k-1})^T - x^{k-1})]^T \cdot [x - x^k] \geq 0, \quad \forall x \in X. \tag{60}$$

Step 3 Convergence Verification. If $| x^k - x^{k-1} | \leq \epsilon$, for $\epsilon > 0$, a prespecified tolerance, then stop. Otherwise, set the iteration index to $k=k+1$ and return to Step 1.

In the context of the above SPE model, problem (59) can be solved for all supply and demand market pairs (i,j); $i=1,\ldots,I$; $j=1,\ldots,J$ by setting

$$\bar{Q}_{ij}^{k-1} = \max\{0, \gamma((-\pi_i(s^{k-1}) - \bar{c}_{ij})(1 + \tau_{ij}) + \rho_j(d^{k-1})) + Q_{ij}^{k-1}\}. \tag{61}$$

and problem (60) can be solved for all supply and demand market pairs *(i,j)*; $i=1,\ldots,I$; $j=1,\ldots,J$ by setting

$$Q_{ij}^{k} = \max\{0, \gamma((-\pi_i(\bar{s}^{k-1}) - \bar{c}_{ij})(1 + \tau_{ij}) + \rho_j(\bar{d}^{k-1})) + Q_{ij}^{k-1}\}. \qquad (62)$$

The derivation of (61) and (62) as closed-form solutions to the approximate and simpler problem $VI(F^k, X)$ is presented in Appendix 1. Note that because the values in the vector (s,Q,d) are given by a previous step for both (61) and (62), evaluation of the expressions is computationally simple even for nonlinear supply and demand price functions with asymmetric cross-price terms. Computing values for (61) and (62) thus requires no optimization techniques. Rather, a series of assignment statements evaluated iteratively in a suitable programming language suffices to compute the solution to the SPE problem.

Given expressions (61) and (62), all of the $I \times J$ commodity shipments can be solved simultaneously at each iteration. This suggests that an "ideal" computer architecture for the solutions of such problems may be one in which there are as many processors as there are pairs of markets, that is, a massively parallel architecture. This issue is investigated computationally in Nagurney *et al.* (1995a).

Nagurney *et al.* (1995b) also show that the MPM above converges to the solution of the variational inequality (17), provided a solution exists, if $\pi(s)$ and $-\rho(d)$ are each strongly monotone in s and d, respectively, with bounded first-order derivatives, and if tariff rates τ_{ij} are finite for all *(i,j)* pairs. Thus, convergence of the MPM is guaranteed for nonlinear supply and demand price functions (under monotonicity conditions) with bounded first-order derivatives (to satisfy Lipschitz continuity), but it is not proven for nonlinear price functions with non-bounded derivatives. (An example of the latter would be the constant elasticity of substitution function often used in empirical SPE and general equilibrium models. The algorithm may, nevertheless, converge in this case.) The computational simplicity of these expressions (even for nonlinear $\pi(s)$ and $\rho(d)$) and the global convergence of the algorithm for characteristics commonly assumed for $\pi(s)$ and $\rho(d)$ are the principal advantages of the variational inequality approach to solving SPE models with *ad valorem* tariffs.

The computational procedure for the SPE problem incorporating activity analysis and *ad valorem* tariffs is similar to that just presented for the SPE problem with continuous product supply functions. The modified projection method uses closed-form expressions such as

$$\bar{s}_{kij}^{k-1} = \max \{ 0, \gamma [(-\sum_{k} (a_{klij} \cdot \omega_{ki}(x^{k-1})) - c_{ij}(Q^{k-1})) \cdot (1 + \tau_{ij}) + \rho_j(d_j^{k-1})] + s_{kij}^{k-1} \}, \quad (63)$$

and

$$s_{kij}^{k} = \max \{ 0, \gamma [(-\sum_{k} (a_{klij} \cdot \omega_{ki}(\bar{x}^{k-1})) - c_{ij}(\bar{Q}^{k-1})) \cdot (1 + \tau_{ij}) + \rho_j(\bar{d}^{k-1})] + s_{kij}^{k-1} \}. \quad (64)$$

The MPM above converges to the solution of the variational inequality (45), provided a solution exists, if $\omega(x)$, $c(Q)$, and $\rho(d)$ are each strongly monotone in x, Q, and d, respectively, with bounded first-order derivatives, and if tariff rates τ_{ij} are finite for all (i,j) pairs.

Numerical Examples of the SPE Model with *Ad Valorem* Tariffs and Incorporating Activity Analysis

To clarify the application of the modified projection method to variational inequalities of the form in (59) and (60), consider a trivial example for two supply and demand markets ($I=2$, $J=2$) with linear, separable supply and demand price functions. Suppose that the relevant supply and demand price functions, transportation costs, and tariff rates are given by:

$$\begin{aligned}
\pi_1(s_1) &= 10 + 1s_1, & \bar{c}_{11} &= 1, & \tau_{11} &= 0.0,\\
\pi_2(s_2) &= 15 + 0.5s_2, & \bar{c}_{12} &= 2, & \tau_{12} &= 0.5,\\
\rho_1(d_1) &= 25 - 1d_1, & \bar{c}_{21} &= 2, & \tau_{21} &= 0.25,\\
\rho_2(d_2) &= 30 - 0.5d_2, & \bar{c}_{22} &= 1, & \tau_{22} &= 0.0.
\end{aligned} \quad (65)$$

Expressions representing (59) from Step 1 can be written for the four *(i,j)* pairs in this simple model as follows:

$$
\begin{aligned}
\bar{Q}_{11}^{0} &= \max\ \{0,\ \gamma((-10\ -\ 1s_1^0\ -\ 1)(1\ +\ 0)\ +\ 25\ -\ 1d_1^0)\ +\ Q_{11}^0\} \\
\bar{Q}_{12}^{0} &= \max\ \{\ 0,\ \gamma((-10\ -\ 1s_1^0\ -\ 2)(1\ +\ 0.5)\ +\ 30\ -\ 0.5d_2^0)\ +\ Q_{12}^0\} \\
\bar{Q}_{21}^{0} &= \max\ \{\ 0,\ \gamma((-15\ -\ 0.5s_2^0\ -\ 2)(1\ +\ 0.25)\ +\ 25\ -\ 1d_1^0)\ +\ Q_{21}^0\} \\
\bar{Q}_{22}^{0} &= \max\ \{\ 0,\ \gamma((-15\ -\ 0.5s_2^0\ -\ 1)(1\ +\ 0)\ +\ 30\ -\ 0.5d_2^0)\ +\ Q_{22}^0\}.
\end{aligned}
\tag{66}
$$

Quantities supplied and demanded for each market can be assigned based on the conservation of flow restrictions (3) and (5), or

$$
\bar{s}_1^{\,0} = \bar{Q}_{11}^0 + \bar{Q}_{12}^0; \quad \bar{s}_2^0 = \bar{Q}_{21}^0 + \bar{Q}_{22}^0; \quad \bar{d}_1^0 = \bar{Q}_{11}^0 + \bar{Q}_{21}^0; \quad \bar{d}_2^0 = \bar{Q}_{12}^0 + \bar{Q}_{22}^0. \tag{67}
$$

Similarly, expressions for (60) from Step 2 can be written for the four *(i,j)* pairs as:

$$
\begin{aligned}
Q_{11}^{1} &= \max\ \{0,\ \gamma((-10\ -\ 1\bar{s}_1^0\ -\ 1)(1\ +\ 0)\ +\ 25\ -\ 1\bar{d}_1^0)\ +\ Q_{11}^0\} \\
Q_{12}^{1} &= \max\ \{\ 0,\ \gamma((-10\ -\ 1\bar{s}_1^0\ -\ 2)(1\ +\ 0.5)\ +\ 30\ -\ 0.5\bar{d}_2^0)\ +\ Q_{12}^0\} \\
Q_{21}^{0} &= \max\ \{\ 0,\ \gamma((-15\ -\ 0.5\bar{s}_2^0\ -\ 2)(1\ +\ 0.25)\ +\ 25\ -\ 1\bar{d}_1^0)\ +\ Q_{21}^0\} \\
Q_{22}^{0} &= \max\ \{\ 0,\ \gamma((-15\ -\ 0.5\bar{s}_2^0\ -\ 1)(1\ +\ 0)\ +\ 30\ -\ 0.5\bar{d}_2^0)\ +\ Q_{22}^0\}.
\end{aligned}
\tag{68}
$$

Supply and demand quantities are again determined using (3) and (5), or

$$
s_1^1 = Q_{11}^1 + Q_{12}^1; \quad s_2^1 = Q_{21}^1 + Q_{22}^1; \quad d_1^1 = Q_{11}^1 + Q_{21}^1; \quad d_2^1 = Q_{12}^1 + Q_{22}^1. \tag{69}
$$

These expressions can be easily computed in spreadsheet packages (or with assignment statements in modeling packages such as GAMS (Brooke *et al.*, 1992)) without the use of optimization algorithms. Setting $\gamma=0.1$, the initial values of all supply, shipment, and demand quantities, (s^0,Q^0,d^0), in Step 0 to zero,[2] and using a convergence tolerance of $\epsilon=0.001$, the modified projection algorithm returned a solution in 78 iterations. This solution was as follows:

$$\begin{array}{llllll} s_1=7.0 & Q_{11}=7.0 & Q_{12}=0.0 & d_1=7.0 & \pi_1=17.0 & \rho_1=18.0 \\ s_2=14.0 & Q_{21}=0.0 & Q_{22}=14.0 & d_2=14.0 & \pi_2=22.0 & \rho_2=23.0\ . \end{array} \tag{70}$$

Imposition of the *ad valorem* tariffs in this example eliminates a positive inter-market trade flow, Q_{12}, existing in the absence of tariffs. Note that nonlinear supply and demand price functions and (or) asymmetric cross-price terms can be incorporated into the closed-form expressions without additional computational complexity. In addition, the size of the problem can be expanded in a straightforward manner by increasing the number of expressions such as (66) through (69) to account for more (i,j) pairs.

As an additional demonstration of applying the MPM to the SPE problem with *ad valorem* tariffs formulated as a variational inequality, we solved the two-good, three-region SPE model from Takayama and Judge (1971, p. 272). This problem specified linear, nonseparable supply and demand price functions and non-discriminatory *ad valorem* tariffs (the linearity of $\pi(s)$ and $\rho(d)$ allowed the problem to be solved with linear complementarity programming). The closed-form expressions of the modified projection method are more numerous in this case, and thus are not presented here. For $\epsilon=.00001$ and $\gamma=0.1$, the modified projection method converged in 2,707 iterations. This problem was solved using GAMS, and a more detailed description of the problem and the GAMS code to solve it are presented in Appendix 2.

---

[2] Initial values of zero for all variables are commonly used to initialize the MPM. However, it is sometimes necessary to select non-zero initial values to achieve the non-zero solution.

Nagurney *et al.* (1995a) propose an evaluation of the variational inequality solution based on the difference between the computed solution and the true underlying equilibrium conditions. The *average error* between the computed solution and a true equilibrium is defined as:

$$Average\ Error = \frac{100}{p} \sum_{ij} \frac{|(\pi_i + \bar{c}_{ij})(1 + \tau_{ij}) - \rho_j|}{(\pi_i + \bar{c}_{ij})(1 + \tau_{ij})}, \tag{71}$$

where p is the number of market pairs (i,j) for which $Q^{*}_{ij} > 0$. The *maximum error* between the computed solution and the equilibrium conditions is defined as:

$$Maximum\ Error = \max_{ij} \frac{100\,|(\pi_i + \bar{c}_{ij})(1 + \tau_{ij}) - \rho_j|}{(\pi_i + \bar{c}_{ij})(1 + \tau_{ij})}. \tag{72}$$

By these measures, our variational inequalities formulation performed very well, with an average error of 0.0004% and a maximum error of 0.001% from the true equilibrium conditions (15).

We now present an example incorporating activity analysis. Consider a problem with $K=2$, $L=2$, $I=2$ and $J=2$, where the input supply, product demand, transport cost functions, and *ad valorem* tariff rates are given by

$$\begin{array}{llll} \omega_{11} = 1(x_{11})^2, & \bar{c}_{11} = 1, & \rho_1 = 100 - 5d_1, & \tau_{11} = 0.0, \\ \omega_{12} = 2(x_{12})^2, & \bar{c}_{12} = 2, & \rho_2 = 50 - 3d_2, & \tau_{12} = 0.0, \\ \omega_{21} = 0.5(x_{21})^2, & \bar{c}_{21} = 2, & & \tau_{21} = 0.3, \\ \omega_{22} = 1(x_{22})^2, & \bar{c}_{22} = 1, & & \tau_{22} = 0.0, \end{array} \tag{73}$$

and the input requirements for production of the good are given by

$$\{ a_{klij} \} = \begin{bmatrix} & (l,i,j)= & & & & & & & \\ & 111 & 112 & 121 & 122 & 211 & 212 & 221 & 222 \\ k= & & & & & & & & \\ 1 & .5 & .5 & .5 & .5 & 2 & 2 & 2 & 2 \\ 2 & 1 & 1 & 1 & 1 & .5 & .5 & .5 & .5 \end{bmatrix} . \tag{74}$$

Eight expressions for the s_{lij}^{t-1} are necessary in this problem, and are defined as in (63). The values of x_{ki}^{t-1}, and d_j^{t-1} necessary in the closed-form expressions are calculated using the values of s_{lij}^{t-1} in the conservation of flow constraints (37). Values for s_{lij}^{t} are calculated analogously. With $\gamma = 0.01$, and an error tolerance of $\epsilon = 0.00001$, the modified projection method required 180 iterations to converge to the optimal solution of

$$\begin{array}{llll} x_{11}=4.421 & x_{21}=8.843 & & \\ x_{12}=2.697 & x_{22}=5.394 & & \\ s_{111}=8.843 & s_{112}=0.000 & s_{211}=0.000 & s_{212}=0.000 \\ s_{121}=1.182 & s_{122}=4.212 & s_{221}=0.000 & s_{222}=0.000 \\ Q_{11}=8.843 & Q_{12}=0.000 & Q_{21}=1.182 & Q_{22}=4.212 \\ d_1=10.025 & d_2=4.212 & & \\ \omega_{11}=19.550 & \omega_{21}=39.099 & \omega_{12}=14.546 & \omega_{22}=29.092 \\ \rho_1=49.874 & \rho_2=37.365\ . & & \end{array} \tag{75}$$

The GAMS code for this problem is provided in Appendix 2.

Note that with this problem, nonlinear functional forms for ρ can be incorporated into the closed-form expression without additional computational complexity. The number of supply markets, demand markets, and production activities could be increased by simply adding additional closed-form expressions. Increasing the number of resources k in the problem does not increase the number of expressions, only the number of terms in each expression. Increasing the number of goods in the problem requires an additional subscript to distinguish among goods, and is therefore straightforward.

Concluding Comments

In this paper, we describe two SPE models with discriminatory *ad valorem* tariffs and apply the methodology of the theory of variational inequalities for their formulation and computation. The motivation for this work originates, in part, from the recent emphasis on tariffication as a means of trade liberalization. We propose an algorithm, the modified projection method, which is proven to converge for our models under nonlinear (and strongly monotone) supply and demand price functions with bounded first-order derivatives and finite *ad valorem* tariff rates. However, the algorithm may, nevertheless, converge to the equilibrium conditions even when these conditions are not satisfied.

A notable feature of the algorithm is that for the specified problems, commodity shipments may be computed using a simple, closed-form expression at each iteration rather than optimization algorithms. This is true even for problems incorporating nonlinear supply and demand price functions and nonlinear, nonseparable transportation cost functions. As a result, additional *(i,j)* market pairs can be included simply by increasing the number of closed-form expressions.

Our method of solving SPE models with discriminatory *ad valorem* tariffs is applicable to modeling trade agreements such as NAFTA and GATT in which tariffication plays an important role. However, other trade and domestic policies and differences in production technologies are also determinants of international trade patterns. Nagurney (1993) provides a detailed treatment of individual policy instruments important to trade analysis (e.g., quotas and price restrictions). However, to date there appear to be no empirical applications of VI methods that simultaneously incorporate all relevant domestic and trade policies. Thus, incorporating a broader range of policies and technologies into a VI framework including *ad valorem* tariffs is a logical extension of the models presented here.

In addition, further research to explore the theoretical and empirical convergence aspects of the various methods for calculating solutions to SPE problems, and comparisons of the computational efficiency for SPE problems with certain characteristics would be a significant contribution to the applied economist's use of SPE models for analysis of trade.

References

Brooke, A., D. Kendrick, and A. Meeraus. *GAMS: A Users Guide, Release 2.25.* San Francisco: The Scientific Press, 1992.

Cottle, R. W., G. J. Habetler, and C. E. Lemke, "Quadratic Forms Semidefinite over Convex Cones," in H. W. Kuhn, ed., *Proceedings of the Princeton Symposium on Mathematical Programming*, 551-565, Princeton: Princeton University Press, 1970.

Cramer, G. L., E. J. Wailes, J. M. Goroski, and S. S. Phillips. "The Impact of Liberalizing Trade on the World Rice Market: A Spatial Model Including Rice Quality," Fayetteville, Arkansas: Arkansas Agricultural Experiment Station, 1991. (Special Report 153)

Dafermos, S. "An Iterative Scheme for Variational Inequalities," *Mathematical Programming*, 26(1983):40-47.

Florian, M., and M. Los. "A New Look at Static Spatial Price Equilibrium Models," *Regional Science and Urban Economics*, 12(1982):579-597.

Ginsburgh, V. A. and L. Van der Heyden. "On Extending the Negishi Approach to Computing Equilibria: the Case of Government Price Support Policies," *Journal of Economic Theory*, 44(1988):168-78.

Harker, P. T. *Lectures on Computation of Equilibria with Equation-Based Methods*, Louvain-la-Neuve, Belgium: CORE, 1993. (CORE Lecture Series)

Holland, F. D. and J. A. Sharples. "World Wheat Trade: Implications for U.S. Exports," Department of Agricultural Economics, Purdue University, West Lafayette, IN, November 1984. (Staff Paper No. 84-20)

Judge, G. G. and T. Takayama, eds. *Studies in Economic Planning Over Space and Time*, Amsterdam: North-Holland, 1973.

Kinderlehrer, D. and G. Stampacchia. *An Introduction to Variational Inequalities.* New York: Academic Press, 1980.

Korpelevich, G. M. "The Extragradient Method for Finding Saddle Points and Other Problems," *Matekon*, 13(1977):35-49.

Kuhn, H. W. and J. G. MacKinnon. "Sandwich Method for Finding Fixed Points," *J. of Optimization Theory and Applications*, 17(1975):189-204.

Mathiesen, L. "Computation of Economic Equilibria by a Sequence of Linear Complementarity Problems," *Operations Research*, 33(1985):144-162.

Nagurney, A. *Network Economics: A Variational Inequality Approach.* Boston, MA: Kluwer Academic Publishers, 1993.

Nagurney, A., C. F. Nicholson, and P. M. Bishop. "Massively Parallel Computation of Large-scale Spatial Price Equilibrium Models with Discriminatory Ad Valorem Tariffs," *The Annals of Operations Research*, Special Issue on Computational Economics, forthcoming 1995(a).

Nagurney, A., C. F. Nicholson, and P. M. Bishop. "Spatial Price Equilibrium Models with Discriminatory Ad Valorem Tariffs: Formulation and Comparative Computation Using Variational Inequalities," in J. C. J. M. van den Bergh, P. Nijkamp and P. Rietveld (eds.), *Recent Advances in Spatial Equilibrium Modelling: Methodology and Applications*, Heidelberg: Springer-Verlag, forthcoming, 1995(b).

Preckel, P. V. "Alternative Algorithms for Computing Economic Equilibria," *Mathematical Programming Study*, 23(1985):163-172.

Samuelson, P. A. "Spatial Price Equilibrium and Linear Programming," *American Economic Review*, 42(1952):283-303.

Takayama, T. and G. G. Judge. *Spatial and Temporal Price and Allocation Models.* Amsterdam: North-Holland, 1971.

Thompson, R. L. "Spatial and Temporal Equilibrium Agricultural Models," in *Quantitative Methods for Market-Oriented Economic Analysis Over Space and Time*, W. C. Labys, T. Takayama, and N. D. Uri, eds., Aldershot, England: Avebury, 1989.

APPENDIX 1

Derivation of the Closed-form Expressions (61) and (62) to Compute the Solution to the SPE problem

Following Harker (1993), we begin with the following definition.

Definition A1. Let X be a non-empty, closed and convex subset of R^n, and let G be a n x n symmetric positive definite matrix. The *projection under the G-norm* of a point $y \in R^n$ onto the set X, denoted by $\pi_{G,X}(y)$ is defined as the solution to the following mathematical programming problem

$$\min \ \frac{1}{2} \|y - x\|_G^2 \ , \ s.t. \ \ x \in X, \qquad \text{(A1.1)}$$

where

$$\|x\|_G = (x'Gx)^{1/2} \qquad \text{(A1.2)}$$

In other words, the projection $\pi_{G,X}(y)$ is the vector in the set X that is closest to y under the G-norm.

We can now use Definition A1 to establish the following fixed-point formulation of a variational inequality problem.

Proposition A1. Let X be a non-empty, closed, convex subset of R^n, and let G be a n x n symmetric positive definite matrix. Then, x^* solves the problem $VI(F, X)$ if and only if

$$x^* = \pi_{G,X}(x^* - G^{-1}F(x^*)); \qquad \text{(A1.3)}$$

i.e., if and only if x^* is a fixed point of the mapping $H{:}R \rightarrow R$ defined by

$$H(x) = \pi_{G,X}(x - G^{-1}F(x)). \qquad \text{(A1.4)}$$

When $X \equiv R^n_+$, $H(x)$ takes the form

$$H(x) = \max(0, x - F(x)) . \tag{A1.5}$$

Recall that a general approach to solving *VI(F, X)* consists of creating a sequence $\{x^k\} \subseteq X$ such that x^{k+1^*} solves an approximate and computationally simpler problem *VI*(F^k, *X*)

$$F^k(x^{k+1^*})^T(x - x^{k+1^*}) \geq 0, \ \forall \ x \in X, \tag{A1.6}$$

where $F^k(x)$ is some approximation to *F(x)*. When $F^k(x)$ is a linear approximation, the form $F^k(x)$ is

$$F^k(x^{k+1}) = F(x^k) + A(x^k)(x^{k+1} - x^k), \tag{A1.7}$$

where $A(x^k)$ is a *n x n* matrix. For the projection method, we choose *G*, a *n x n* fixed, symmetric, positive definite matrix as $A(x^k)$. So the approximate problem *VI*(F^k, *X*) can be written as find x^{k+1^*} such that

$$F^k(x^{k+1^*})^T(x - x^{k+1^*}) \geq 0, \ \forall \ x \in X, \tag{A1.8}$$

or, substituting our linear approximation of $F^k(x)$ in (A1.7), find x^{k+1^*} such that

$$[F(x^k) + G(x^{k+1^*} - x^k)]^T(x - x^{k+1}) \ \geq \ 0, \ \forall \ x \in X. \tag{A1.9}$$

From (A1.3) and (A1.4), we know that x^{k+1^*} solves the problem *VI*(F^k, *X*) if x^{k+1^*} is a fixed point of the mapping

$$H(x^{k+1^*}) = \pi_{G,X}(x^{k+1^*} - G^{-1}F(x^{k+1^*})), \tag{A1.10}$$

or, substituting the linear approximation of $F^k(x)$ in (A1.10),

$$H(x^{k+1}) = \pi_{G,X}(x^{k+1} - G^{-1}[F(x^k)+G(x^{k+1} - x^k)])$$

$$= \pi_{G,X}(x^{k+1} - G^{-1}F(x^k) - x^{k+1} + x^k \qquad \text{(A1.11)}$$

$$= \pi_{G,X}(-G^{-1}F(x^k) + x^k)$$

From (A1.5) we know that this projection has the form

$$\max\ (0, -G^{-1}F(x^k) + x^k). \qquad \text{(A1.12)}$$

Let

$$G = L \cdot I_n, \qquad \text{(A1.13)}$$

where L is the Lipschitz continuity constant in (30) and (36). Thus, G is a $n\ x\ n$ fixed, symmetric, positive definite matrix for $L > 0$, and

$$G^{-1} = (\frac{1}{L}) \cdot I_n = \gamma \cdot I_n. \qquad \text{(A1.14)}$$

Thus, the form of the solution to the approximate problem $VI(F^k, X)$ is

$$x^{k+1} = \max\ (0, -\gamma \cdot F(x^k) + x^k). \qquad \text{(A1.15)}$$

Appendix 2

GAMS Code for Solving the Numerical Examples

The Takayama and Judge Problem Formulated as a VI Using the Modified Projection Method

```
$TITLE VITAK.GMS  CFN/PMB  9-27-94
$OFFSYMXREF OFFSYMLIST OFFUELLIST OFFUELXREF OFFUPPER
*
*  MODEL TO REPLICATE THE TAKAYAMA-JUDGE 3 REGION, 2 PRODUCT
*  TRADE PROBLEM WITH AD VALOREM TARIFFS USING VARIATIONAL
*  INEQUALITIES.  THE MODIFIED PROJECTION METHOD IS USED.
*  SEE TAKAYAMA AND JUDGE, 1971 P. 267-272.
*  ALPHA DENOTES SUPPLY FUNCTION PARAMETERS
*  BETA DENOTES DEMAND FUNCTION PARAMETERS
*  I=1,2,3 SUPPLY REGIONS
*  J=1,2,3 DEMAND REGIONS
*  K=A,B PRODUCTS
*
 SETS I regions          /1*3/
      K products         /A, B/
      ITER iteration     /1*3000/
 ALIAS (I,J);

 TABLE PARAMS(I,K,*)  supply and demand function parameters

           ALPHA0   ALPHA1   ALPHA2    BETA0     BETA1     BETA2
     1.A   4.6150   0.1003   -0.0067   21.6080   -0.1005   -0.0050
     1.B   3.8462   0.0669   -0.0033   32.1608   -0.1005   -0.0100
     2.A   2.3824   0.0501   -0.0020   23.4694   -0.2041   -0.0204
     2.B   2.3524   0.0400   -0.0010   34.6939   -0.2041   -0.0408
     3.A   4.6154   0.1003   -0.0067   21.6981   -0.1258   -0.0063
     3.B   3.8462   0.0669   -0.0033   27.1698   -0.1006   -0.0126
```

TABLE TCOST(I,J,K) unit transport costs from I to J

	1.A	1.B	2.A	2.B	3.A	3.B
1	0	0	2	3	2	3
2	2	3	0	0	1	2
3	2	3	1	2	0	0

TABLE TARIFF(I,J,K) ad valorem import tariffs imposed by region J

	1.A	1.B	2.A	2.B	3.A	3.B
1	0	0	0	0	0.3	0.3
2	0.2	0.3	0	0	0	0
3	0.2	0.3	0	0	0	0

```
PARAMETER
      S0(I,K)            supply of product K in region I
      S0BAR(I,K)         supply of product K in region I
      S1(I,K)            supply of product K in region I
      Q0(I,J,K)          shipment of K from region I to region J
      Q0BAR(I,J,K)       shipment of K from region I to region J
      Q1(I,J,K)          shipment of K from region I to region J
      D0(J,K)            demand of product K in region J
      D0BAR(J,K)         demand of product K in region J
      D1(J,K)            demand of product K in region J
      SP(I,K)            supply price of K in region I
      DP(J,K)            demand price of K in region J
      QDIF(I,J,K)        difference between Q0 and Q1
      ERR(I,J,K)         error computation
      COUNT(I,J,K)       count of non-negative Q1's ;

SCALAR
      ITCOUNT            iteration count                    /0/
      GAMMA              inverse of Lipschitz constant      /0.1/
      MAXQDIF            maximum value of QDIF              /1/
      AVGERR             average error                      /0/
      MAXERR             maximum error                      /0/ ;

* INITIALIZING THE RIGHT-HAND-SIDE PARAMETERS

      S0(I,K)      = 0.0;
      D0(J,K)      = 0.0;
      Q0(I,J,K)    = 0.0;
```

```
LOOP(ITER$(MAXQDIF GT 0.00001),
    Q0BAR(I,J,K) $ (ORD(K) EQ 1) = MAX(0,
    GAMMA*((-(PARAMS(I,K,'ALPHA0')+PARAMS(I,K,'ALPHA1')*S0(I,K)
   + PARAMS(I,K,'ALPHA2')*S0(I,K+1)) - TCOST(I,J,K))*(1+TARIFF(I,J,K))
   + (PARAMS(J,K,'BETA0') + PARAMS(J,K,'BETA1')*D0(J,K)
   + PARAMS(J,K,'BETA2')*D0(J,K+1))) + Q0(I,J,K));

    Q0BAR(I,J,K) $ (ORD(K) EQ 2) = MAX(0,
    GAMMA*((-(PARAMS(I,K,'ALPHA0')+PARAMS(I,K,'ALPHA1')*S0(I,K)
   + PARAMS(I,K,'ALPHA2')*S0(I,K-1)) - TCOST(I,J,K))*(1+TARIFF(I,J,K))
   + (PARAMS(J,K,'BETA0') + PARAMS(J,K,'BETA1')*D0(J,K)
   + PARAMS(J,K,'BETA2')*D0(J,K-1))) + Q0(I,J,K));

   S0BAR(I,K) = SUM(J, Q0BAR(I,J,K));
   D0BAR(J,K) = SUM(I, Q0BAR(I,J,K));

    Q1(I,J,K) $ (ORD(K) EQ 1) = MAX(0,
    GAMMA*((-(PARAMS(I,K,'ALPHA0')+PARAMS(I,K,'ALPHA1')*S0BAR(I,K)
   + PARAMS(I,K,'ALPHA2')*S0BAR(I,K+1)) - TCOST(I,J,K))*(1+TARIFF(I,J,K))
   + (PARAMS(J,K,'BETA0') + PARAMS(J,K,'BETA1')*D0BAR(J,K)
   + PARAMS(J,K,'BETA2')*D0BAR(J,K+1))) + Q0(I,J,K));

    Q1(I,J,K) $ (ORD(K) EQ 2) = MAX(0,
    GAMMA*((-(PARAMS(I,K,'ALPHA0')+PARAMS(I,K,'ALPHA1')*S0BAR(I,K)
   + PARAMS(I,K,'ALPHA2')*S0BAR(I,K-1)) - TCOST(I,J,K))*(1+TARIFF(I,J,K))
   + (PARAMS(J,K,'BETA0') + PARAMS(J,K,'BETA1')*D0BAR(J,K)
   + PARAMS(J,K,'BETA2')*D0BAR(J,K-1))) + Q0(I,J,K));

   QDIF(I,J,K)   = Q1(I,J,K)-Q0(I,J,K);
   MAXQDIF       = SMAX((I,J,K), ABS(QDIF(I,J,K)));
   ITCOUNT       = ORD(ITER);
   S0(I,K)       = SUM(J, Q1(I,J,K));
   Q0(I,J,K)     = Q1(I,J,K);
   D0(J,K)       = SUM(I, Q1(I,J,K))
);

* QUANTITIES

   S1(I,K)       = S0(I,K);
   Q1(I,J,K)     = Q0(I,J,K);
   D1(J,K)       = D0(J,K);

* PRICES

   SP(I,K)$(ORD(K) EQ 1) = PARAMS(I,K,'ALPHA0') +
   PARAMS(I,K,'ALPHA1')*S1(I,K) + PARAMS(I,K,'ALPHA2')*S1(I,K+1);
   SP(I,K)$(ORD(K) EQ 2) = PARAMS(I,K,'ALPHA0') +
```

```
    PARAMS(I,K,'ALPHA1')*S1(I,K) + PARAMS(I,K,'ALPHA2')*S1(I,K-1);
    DP(J,K)$(ORD(K) EQ 1) = PARAMS(J,K,'BETA0') +
    PARAMS(J,K,'BETA1')*D1(J,K) + PARAMS(J,K,'BETA2')*D1(J,K+1);
    DP(J,K)$(ORD(K) EQ 2) = PARAMS(J,K,'BETA0') +
    PARAMS(J,K,'BETA1')*D1(J,K) + PARAMS(J,K,'BETA2')*D1(J,K-1);

* ERROR COMPUTATIONS

    ERR(I,J,K)$(Q1(I,J,K) NE 0) =
    100*(ABS((SP(I,K)+TCOST(I,J,K))*(1+TARIFF(I,J,K))-DP(J,K))/
    (SP(I,K)+TCOST(I,J,K))*(1+TARIFF(I,J,K)));

 COUNT(I,J,K)      = 1$(Q1(I,J,K) NE 0);

 AVGERR            = SUM((I,J,K), ERR(I,J,K))/(SUM((I,J,K),COUNT(I,J,K)));

 MAXERR            = SMAX((I,J,K), ABS(ERR(I,J,K)));

 DISPLAY MAXQDIF, ITCOUNT, S1, Q1, D1, SP, DP, ERR, AVGERR, MAXERR;
```

The SPE Model Incorporating Activity Analysis Formulated as a VI Using the Modified Projection Method

```
$TITLE VIAA.GMS  CFN/PMB  9/27/94
$OFFUPPER OFFSYMXREF OFFSYMLIST OFFUELLIST OFFUELXREF
*
* A SIMPLE TRADE MODEL INCORPORATING ACTIVITY ANALYSIS AND
* AD VALOREM TARIFFS, AND SOLVED USING VARIATIONAL
* INEQUALITIES.  THERE ARE 2 REGIONS, 2 RESOURCES (INPUTS),
* AND 2 PRODUCTION TECHNOLOGIES.
* THE MODIFIED PROJECTION METHOD IS USED.
* I=1,2 PRODUCTION REGIONS
* J=1,2 CONSUMPTION REGIONS
* K=1,2 RESOURCES (X'S)
* L=1,2 PRODUCTION TECHNOLOGIES TO PRODUCE A SINGLE GOOD
*
 SETS I                  /1*2/
      ITER iterations    /1*1000/ ;
 ALIAS (I,J,K,L);
```

TABLE PARAMS(J,*) linear demand function parameters

	BETA0	BETA1
1	100	-5
2	100	-3

TABLE A(K,L,I,J) input output coefficients

	1.1	1.2	2.1	2.2
1.1	0.5	0.5	0.5	0.5
1.2	2	2	2	2
2.1	1	1	1	1
2.2	0.5	0.5	0.5	0.5

TABLE ACOST(I,K) scale parameter on quadratic input cost functions

	1	2
1	1	0.5
2	2	1

TABLE TARIFF(I,J) ad valorem import tariffs imposed by region J

	1	2
1	0	0
2	0.3	0

TABLE TCOST(I,J) unit transport costs from I to J

	1	2
1	1	2
2	2	1

PARAMETER

```
X0(I,K)        resource K used in region I
X0BAR(I,K)     resource K used in region I
X1(I,K)        resource K used in region I
S0(L,I,J)      production in region I by technology L for region J
S0BAR(L,I,J)   production in region I by technology L for region J
```

```
      S1(L,I,J)      production in region I by technology L for region J
      STOTAL(I)      total production in region I
      Q1(I,J)        shipments from region I to region J
      D0(J)          demand in region J
      D0BAR(J)       demand in region J
      D1(J)          demand in region J
      XP(I,K)        price of resource K in region I
      SP(L,I,J)      supply price of the good in I produced by L and shipped to J
      DP(J)          demand price of the good in region J
      SDIF(L,I,J)    difference between S0 and S1 ;

SCALAR
      ITCOUNT        iteration count                       /0/
      GAMMA          inverse of Lipschitz constant         /0.01/
      MAXSDIF        maximum value of SDIF                 /1/ ;

* INITIALIZING THE RIGHT-HAND-SIDE PARAMETERS

      X0(I,K)        = 0;
      D0(J)          = 0;
      S0(L,I,J)      = 0;

LOOP(ITER$(MAXSDIF GT 0.00001),
       S0BAR(L,I,J) = MAX(0, GAMMA*((-SUM(K, A(K,L,I,J)*ACOST(I,K)
       *(X0(I,K)**2)) - TCOST(I,J))*(1+TARIFF(I,J))
      + PARAMS(J,'BETA0') + PARAMS(J,'BETA1')*D0(J)) + S0(L,I,J));

      X0BAR(I,K)     = SUM((L,J), A(K,L,I,J)*S0BAR(L,I,J));
      D0BAR(J)       = SUM((I,L), S0BAR(L,I,J));

       S1(L,I,J) = MAX(0, GAMMA*((-SUM(K, A(K,L,I,J)*ACOST(I,K)
       *(X0BAR(I,K)**2)) - TCOST(I,J))*(1+TARIFF(I,J))
      + PARAMS(J,'BETA0') + PARAMS(J,'BETA1')*D0BAR(J)) + S0BAR(L,I,J));

      SDIF(L,I,J)    = S1(L,I,J)-S0(L,I,J);
      MAXSDIF        = SMAX((L,I,J), ABS(SDIF(L,I,J)));
      ITCOUNT        = ITCOUNT+1;
      S0(L,I,J)      = S1(L,I,J);
      D0(J)          = SUM((I,L), S1(L,I,J));
      X0(I,K)        = SUM((L,J), A(K,L,I,J)*S1(L,I,J))
);

* QUANTITIES

      X1(I,K)        = X0(I,K);
      S1(L,I,J)      = S0(L,I,J);
      STOTAL(I)      = SUM((L,J), S1(L,I,J));
```

```
      Q1(I,J)      = SUM(L, S1(L,I,J));
      D1(J)        = D0(J);

* PRICES

      XP(I,K)      = X1(I,K)**2;
      SP(L,I,J)    = ACOST(I,K)*SUM(K, A(K,L,I,J)*XP(I,K));
      DP(J)        = PARAMS(J,'BETA0') + PARAMS(J,'BETA1')*D1(J);

 DISPLAY MAXSDIF, ITCOUNT, X1, S1, STOTAL, Q1, D1, XP, SP, DP;
```

OTHER A.R.M.E. WORKING PAPERS
(Formerly A.E. Working Papers)