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Stock-Dependent Cost**

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# Optimal Depletion of Non-Renewable Resources under Linear Homogeneous Stock-Dependent Cost

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## ABSTRACT

Models of optimal depletion in which marginal extraction cost rises with depletion, although more realistic than constant cost models, appear seldom in the literature. A reasonable explanation for this is that, in the absence of considerable simplifying assumptions, problems of this nature are difficult, if not impossible, to solve analytically.

If the cost of extraction is assumed to be homogeneous of degree one in output and remaining reserves, and if prices are assumed parametric, many of the difficulties can be resolved. The solution to a discrete-time, finite horizon version of such a problem is obtained through dynamic programming, and illustrated graphically. Analysis of the solution indicates that the reduction of future stock value which results from cost increases can be interpreted as a different type of user cost, or opportunity cost. The "profit" earned in each period is shown to be just sufficient to compensate for the reduction in the value of the mineral asset caused by its extraction activities. The operation of an "r-percent rule" at the level of the individual deposit is demonstrated.

Comparative analysis is done, demonstrating the effects on the optimal extraction path of single- and multi-period price changes. The results highlight the fact that, while reserves defined in a physical sense are exogenous, reserves defined in an economic sense, dependent on prices and cost, are endogenous. A change in the prices received by the firm alters not only the pattern of production, but also the amount of stock which is ultimately recovered.

In a recent paper, Cairns (1990) suggested that there are really two branches in the literature on non-renewable resources. In contrast to "the more macro preoccupations of the Hotelling tradition" he suggested that micro considerations seem to have been considered of minor importance. This may explain why some of the interesting microeconomic aspects of non-renewable resource problems, such as extraction costs, are often oversimplified.

Models of extraction from a deposit the quality of which degrades or declines with depletion have been rare. This may be due to economists' quest for theoretical results which apply in the most general of cases. This search may have been discouraged by the conclusion of Smith (1974). When discussing a problem in which "recovery becomes more costly...as mines and oil wells get deeper", he concluded, somewhat pessimistically (an ironic mood in view of his title), that "an exact characterization of such solution possibilities does not appear to be easily specified."

The few attempts which have been made to deal with extraction cost functions of this type have been confined to statements of optimality conditions. Levhari and Liviatan (1977) and Pindyck (1978) have derived modifications to Hotelling's rule which describe the equation of the price path when costs are stock-dependent. Fisher (1981) presented a similar result; closer to a microeconomic treatment of the problem in that it focusses on the dynamics of the value of reserves, rather than on those of price.

Methods for obtaining concrete solutions to this type of problem have been more elusive. A condition describing an optimal solution, valuable as it might be, is quite a different thing from the solution itself. Optimal control, or the maximum principle, can provide no more than these necessary conditions. In order to generate actual solutions, we are forced to use other techniques more suited to the task.

If the problem is framed in discrete time, and if it is characterized by a finite horizon, it is reasonable to attempt to find a solution by means of dynamic programming. When we attempt this, the nature of the difficulties caused by the inclusion of loosely-specified stock-dependent cost functions immediately becomes clear. The usual first order condition characterizing interior values of optimal quantities, familiar to any student of introductory economics, requires that marginal cost equal marginal net revenue. When cost depends on stock, however, the position of the marginal cost curve in any given period cannot be determined without reference to the amount of remaining reserves, which, in turn, requires knowledge of the quantities optimally extracted in earlier periods. Unfortunately, this is unknown, indeed, it is precisely what the economic analyst wishes to find. He is then required to solve for the entire production profile simultaneously, using some sort of numerical optimization procedure.

Although such solutions can now be obtained easily and cheaply, they provide little insight into the qualitative characteristics of the problems solved thereby. Machine-generated answers provide little or no information about why the solution is what it is, or how sensitive it is to changes in parameters, such as prices or the discount rate. Determining the effects on optimal production programs of such changes through repeated computation of numerical solutions is clumsy and time-consuming at best.

What is needed is a way of modelling stock dependent costs which captures the main features of the problem, yet is easily solved and analyzed. In this paper, such a model is outlined, solved, and discussed.

The simplest type of cost function which can be used to capture the effect of stock dependence has only two arguments, current period output and some measure of the amount of stock remaining. For our purposes, such a cost function will suffice if it allows us to specify exactly how marginal cost rises as the stock is depleted.

It should be noted that the arguments of this cost function are not completely independent. The level of stock available in a given period depends on the decision made in the previous period concerning how much of the stock to extract. An increase in current period extraction, then, has two consequences. It raises costs in the current period, and it shifts the cost curve in future periods upwards.

The cost function used in this paper is assumed to be homogeneous of degree one in the two arguments, which implies that these two consequences of extraction are linked in a specific way. This restriction permits the derivation of a variant of the cost function which has as its only argument the proportion of available stock extracted in each period. The benefits of using such a form are several. First, the modified cost function is fixed in place for each period, allowing the problem to be solved using conventional dynamic programming techniques. Second, because cost and revenue are functions of a proportion, rather than an absolute quantity, the diagrams illustrating optimal solutions lend themselves easily to interpretation in terms of values per unit of stock. Third, the relation between the current period level of extraction and the change in cost in the subsequent period can be specified exactly, illustrating clearly the operation of an  $r$ -percent rule at the level of the individual deposit. Fourth, since the functions defining the optimal proportion to extract are functions of price and the discount rate, they may be used to ascertain the effects of changes in these parameters on the optimal extraction path.

The paper is divided into three parts. First, the model is stated and solved. The solution is shown both algebraically and

graphically. Second, the connection between current period extraction and future declines in reserve values due to increased cost is discussed, and the modified r-per-cent rule is given intuitive interpretation. Third, the model is used to analyze the effects of single- and multi-period changes in prices on the entire extraction path. This reveals that, while the physical size of stock remains fixed, the economic size of the reserve is, in fact, endogenous to the model, dependent on output price and cost conditions.

## I. THE MODEL

The general form of the pure depletion problem is:

$$\begin{aligned} & \text{MAX}_{\{Y_t\}} \Pi(P_1, P_2, \dots, P_{T-1}, \delta) \\ & = \sum_{t=0}^{T-1} \rho^t \{R(Y_t) - C(Y_t, S_t)\} \end{aligned}$$

Subject to:

$$S_{t+1} = S_t - Y_t, \quad S_0 \text{ given}$$

where  $R$  is total revenue,  $C$  is total cost,  $Y$  and  $S$  represent production and remaining stock respectively, and  $\rho = (1+\delta)^{-1}$ , where  $\delta$  is the rate of discount used.

To guarantee the existence of a maximum, we also require:

- (a)  $R_Y(Y_t) > 0$
- (b)  $R_{YY}(Y_t) \leq 0$ ,
- (c)  $C_Y(Y_t, S_t) > 0$ ,
- (d)  $C_{YY}(Y_t, S_t) \geq 0$ ,
- (e)  $C_S(Y_t, S_t) \leq 0$ .

Stock-dependent costs imply that inequality (e) is strict.

If the marginal cost of extraction, as well as the total cost, increases as the reserve declines, we also have

$$(f) \quad C_{Y_S}(Y_t, S_t) < 0$$

Let us now impose the restriction that the cost function is homogeneous of degree one in  $Y$  and  $S$  for any constant set of factor prices. (Since the firm is assumed to own the mineral reserve, the sunk cost of its acquisition is not relevant for extraction decisions.) The cost function can be written as the product of a function of the proportion of remaining stock to be extracted and the absolute size of this remaining stock.<sup>1</sup> To see this, we define, for any period  $t$ ,

$$Y_t = \gamma_t S_t$$

Using the properties of linear homogeneous functions, we have

$$\frac{1}{S_t} C(Y_t, S_t) = c\left(\frac{Y_t}{S_t}, 1\right) = c\left(\frac{Y_t}{S_t}\right) = c(\gamma_t)$$

Therefore, we may write

$$C(Y_t, S_t) = c(\gamma_t) \cdot S_t$$

The objective function of a price taking firm facing such costs may now be written as

$$\begin{aligned} & \text{MAX}_{\{\gamma_t\}} \Pi(P_1, P_2, \dots, P_{T-1}, \delta) \\ & = \sum_{t=0}^{T-1} \rho^t \{ [P_t \gamma_t - c(\gamma_t)] S_t \} \end{aligned}$$

The solution is obtained through dynamic programming, proceeding by backward induction. We begin by finding  $\gamma_{T-1}$ , the

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<sup>1</sup> The proposition that such a function might reasonably be applied to an empirical situation can be illustrated with a Cobb-Douglas-type cost function of the form

$$C(Y_t, S_t) = A Y_t^\alpha S_t^{1-\alpha}, \quad A > 0, \alpha > 1$$

The parameter  $\alpha$  describes both the degree to which contemporaneous cost increases with extraction and the extent to which future costs are increased by current production.

proportion of stock to be extracted which maximizes the profit on each unit remaining until the final production period:

$$\text{MAX}_{\{\gamma_{T-1}\}} P_{T-1}\gamma_{T-1} - C(\gamma_{T-1})$$

The first order necessary condition for an interior maximum is the familiar equating of marginal revenue and marginal cost, where these are functions of  $\gamma$ , the proportion extracted:

$$P_{T-1} - C'(\gamma_{T-1}) = 0$$

Because the final period per-unit profit function is concave in  $\gamma_{T-1}$ , this condition is sufficient to define an optimal value of  $\gamma$  provided  $P_{T-1}$  is less than  $c'(1)$ . If  $P_{T-1}$  exceeds  $c'(1)$ , we have a corner solution, and the optimal  $\gamma_{T-1}$  will be one.

Denoting the optimal value of  $\gamma$  by  $\gamma^*$ , profit per unit of stock remaining in the final period is

$$\begin{aligned}\phi_{T-1} &= \frac{\pi_{T-1}^*}{S_{T-1}} = \int_0^{\gamma_{T-1}^*} \{P_{T-1} - c'(Z)\} dZ \\ &= P_{T-1}\gamma_{T-1}^* - C(\gamma_{T-1}^*)\end{aligned}$$

These results are illustrated on the following page. In Figure 1 we see how the optimal  $\gamma$  is determined when  $P_{T-1}$  is less than  $c'(1)$ . The optimal amount to extract is that proportion of remaining reserves at which the gain from increasing the proportion (the price of an assumed infinitely divisible unit) is just equal to the marginal cost of doing so. The value to the firm of each unit remaining at the last period ( $\phi_{T-1}$ ) is shown by the area below the marginal revenue curve (or the price line) and above the marginal cost curve ( $c'(\gamma)$ ). This can be interpreted as the producer's surplus per unit of remaining reserves.

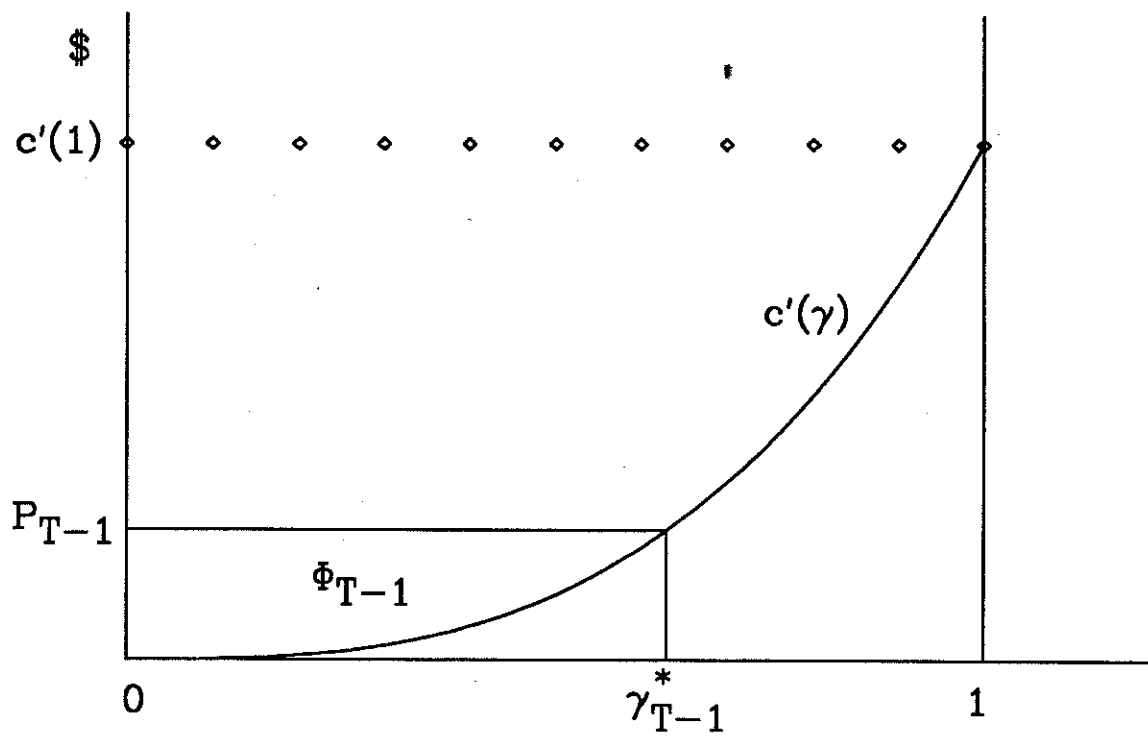
We move next to the determination of  $\gamma_{T-2}$ . This problem is slightly more complicated since each unit left in situ in  $T-2$  has value to the firm, equal to the discounted value of  $\phi_{T-1}$ .

The objective function for period  $T-2$  is:

$$\text{MAX}_{\{\gamma_{T-2}\}} [(P_{T-2}\gamma_{T-2} - C(\gamma_{T-2}))S_{T-2} + \rho\phi_{T-1}S_{T-1}]$$



Figure 1



The stock available in  $T-1$  is equal to the proportion of  $T-2$  stock which is not extracted in  $T-2$ . Therefore, we may rewrite the objective function as

$$[(P_{T-2}\gamma_{T-2} - C(\gamma_{T-2})]S_{T-2} + \rho\phi_{T-1}(1-\gamma_{T-2})S_{T-2}$$

Dividing by  $S_{T-2}$  and rearranging gives

$$(P_{T-2} - \rho\phi_{T-1})\gamma_{T-2} - C(\gamma_{T-2}) + \rho\phi_{T-1}$$

The first order condition for an interior solution<sup>2</sup> is

$$(P_{T-2} - \rho\phi_{T-1}) - C'(\gamma_{T-2}^*) = 0$$

which is sufficient to define the optimal value of  $\gamma_{T-2}$ .

The firm's total profit in period  $T-2$  is

$$\pi_{T-2}^* = \{P_{T-2}\gamma_{T-2}^* - C'(\gamma_{T-2}^*)\}S_{T-2} = \phi_{T-2}S_{T-2}$$

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<sup>2</sup> In this period (as in all periods except the last), it is possible that the optimal value of  $\gamma$  is zero, even when price exceeds marginal cost at some extraction levels. Units will be extracted in the current period only if the gain from doing so is greater than the discounted value of holding them until the next period. There will be no extraction in period  $T-2$  if

$$P_{T-2} \leq \rho\phi_{T-1}$$

Therefore, zero extraction in one or more periods does not in itself imply that the firm has abandoned the mineral deposit. Rather, it may indicate that expected future profit exceeds any possible current period gain from extraction, inducing the firm to wait.

Similarly, if the price in the current period is extremely high, greater profit may be earned by extracting the entire reserve in the current period than by leaving any units in the ground for the future. Algebraically, the condition under which the optimal proportion to extract equals one is

$$P_{T-2} - \rho\phi_{T-1} \geq c'(1)$$

The current value of total reserves available in  $T-2$  is the sum of current period profit and the value of the firm's asset in the form of in situ reserves held for the following period (or the present value of profit to be earned on those reserves):

$$\begin{aligned}\text{Value of } S_{T-2} &= \phi_{T-2} S_{T-2} + \rho \phi_{T-1} S_{T-1} \\ &= \phi_{T-2} S_{T-2} + \rho \phi_{T-1} (1 - \gamma_{T-2}^*) S_{T-2} \\ &= [\phi_{T-2} + \rho \phi_{T-1} (1 - \gamma_{T-2}^*)] S_{T-2}\end{aligned}$$

The expression in square brackets defines the firm's internal value of each unit of reserves available in  $T-2$ .

By expanding  $\phi_{T-2}$ , and rearranging, this may also be written as the sum of the value of all units left until  $T-1$  plus the "surplus" gained from extracting a portion in  $T-2$ :

$$\begin{aligned}\phi_{T-2} + \rho \phi_{T-1} (1 - \gamma_{T-2}^*) &= P_{T-2} \gamma_{T-2}^* - C(\gamma_{T-2}^*) + \rho \phi_{T-1} (1 - \gamma_{T-2}^*) \\ &= [(P_{T-2} - \rho \phi_{T-1}) \gamma_{T-2}^* - C(\gamma_{T-2}^*)] + \rho \phi_{T-1}\end{aligned}$$

The user cost in period  $T-3$  is simply this internal value discounted one period. We will denote by  $\theta_t$  the user cost in period  $t$ . This is one convention which could have been adopted; the other is to represent by a symbol (say  $\lambda_t$ ) the internal value of a unit of reserves in period  $t$ , in which case the user cost in period  $t$  would be  $\rho \lambda_{t+1}$ .

The user cost profile is computed backwards, using the requirement that any units not extracted in the final production period have no value to the firm. We have

$$\begin{aligned}\theta_{T-1} &= 0 \\ \theta_{T-2} &= \rho \phi_{T-1} \\ &= \rho [\phi_{T-1} + (1 - \gamma_{T-1}^*) \theta_{T-1}] \\ \theta_{T-3} &= \rho \phi_{T-2} + \rho^2 (1 - \gamma_{T-1}^*) (1 - \gamma_{T-2}^*) \phi_{T-1} \\ &= \rho [\phi_{T-2} + (1 - \gamma_{T-2}^*) \theta_{T-2}] \quad . . . \\ . . . \theta_{T-k} &= \rho [\phi_{T-k+1} + (1 - \gamma_{T-k+1}^*) \theta_{T-k+1}] \\ &= \sum_{j=1}^{k-1} \rho^j \phi_{T-k+j} \left[ \prod_{i=1}^{j-1} (1 - \gamma_{T-k+i}^*) \right]\end{aligned}$$

The optimal proportion of reserves to extract in each period satisfies

$$\left( P_{T-k} - \sum_{j=1}^{k-1} \rho^j \phi_{T-k+j} \left[ \prod_{i=1}^{j-1} (1 - \gamma_{T-k+i}^*) \right] \right) = c'(\gamma_{T-k}^*)$$

$$\rightarrow (P_{T-k} - \theta_{T-k}) = c'(\gamma_{T-k}^*)$$

The optimal values of  $\gamma_t$ ,  $\phi_t$  and  $\theta_t$ , for  $0 \leq t \leq T-2$  are illustrated in Figure 2. The horizontal axis is again the unit interval, within which the optimal value of  $\gamma_t$  must lie. The quantity  $\theta_t$  is represented by the vertical distance between  $P_t$  and the price net of user cost,  $[P_t - \theta_t]$ . The shaded area in total represents the internal value of reserves held at the beginning of period  $t$ . While this is shown as a two-dimensional area in the figure representing period  $t$ , the same quantity, adjusted by the discount factor, is represented in the corresponding figure for period  $t-1$  by a one-dimensional vertical distance.

The internal value of each unit of reserves, the sum of all shaded areas, can be divided in two ways. The two shaded areas to the left of  $\gamma^*$  add to  $\phi_t$ , the firm's net revenue in period  $t$ . The remaining shaded area represents  $\theta_t(1-\gamma^*)$ , the current period value of that proportion of remaining reserves left in the ground. Alternatively,  $(P_t - \theta_t) - c(\gamma_t)$ , or the "surplus", can be represented by the area shaded with diamonds, and  $\theta_t$ , the value of each unit left in situ, defined by the rectangle with height  $P_t - (P_t - \theta_t)$  and width one.

Having determined the optimal proportions of reserves to be extracted, the optimal quantities themselves may then be calculated as

$$Y_{T-k}^* = \gamma_{T-k}^* S_{T-k}^*$$

where

$$\gamma_{T-k}^* = f(P_{T-k}, P_{T-k+1}, \dots, P_{T-1}, \delta, [c(\gamma)])$$

$$S_{T-k}^* = g(S_0, \gamma_0^*(\cdot), \gamma_1^*(\cdot), \dots, \gamma_{T-k-1}^*(\cdot))$$

$$= S_0 \prod_{j=0}^{T-k-1} (1 - \gamma_j^*(\cdot))$$

The total discounted profit earned from the entire extraction profile can be expressed in two ways. First, it is equal to the sum of each period's income, suitably discounted, where income each period is expressed as  $\phi_t S_t$ , where  $S_t$  in turn is a fraction of  $S_0$ , the original reserve:

$$\begin{aligned}\Pi &= \phi_0 S_0 + \rho \phi_1 (1-\gamma_0^*) S_0 + \rho^2 \phi_2 (1-\gamma_0^*) (1-\gamma_1^*) S_0 + \dots \\ &\quad \dots + \rho^{T-1} \phi_{T-1} \left[ \prod_{i=0}^{T-2} (1-\gamma_i^*) \right] S_0 \\ &= S_0 \left\{ \phi_0 + \sum_{j=1}^{T-1} \rho^j \phi_j \left[ \prod_{i=0}^{j-1} (1-\gamma_i^*) \right] \right\}\end{aligned}$$

Alternatively, total discounted profit is equal to the sum of initial period profit plus the internal value of all units held in the form of in situ reserves for the remainder of the production horizon:

$$\Pi = S_0 \{ \phi_0 (P_0) + [1-\gamma_0^*(P_0)] \cdot \theta_0 (P_1, P_2, \dots, P_{T-1}) \}$$

Total profit is more easily computed using the second expression, since it is simply the sum of two variables the optimal values of which were determined in calculating the series of  $\gamma_t$ 's for the problem. Use of the first expression becomes progressively more cumbersome as  $T$ , the length of the horizon, increases.

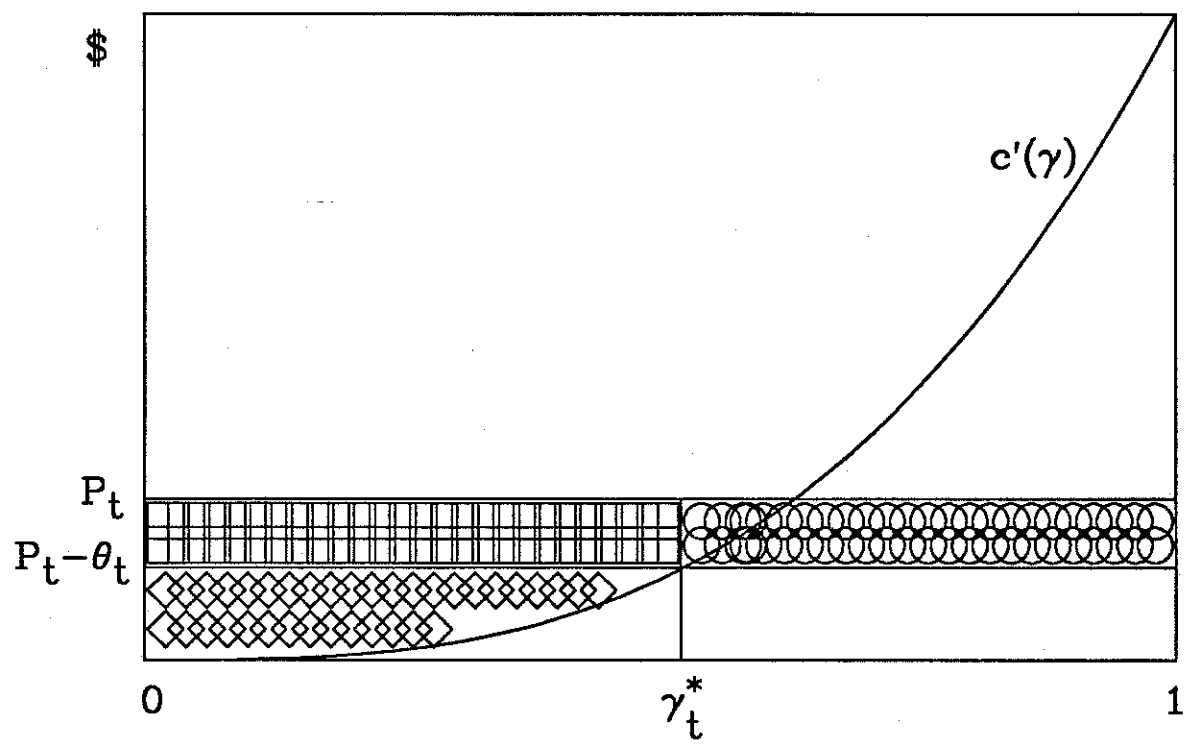
## ANALYSIS AND DISCUSSION

The firm which owns, or has rights to, a stock of a mineral the quality of which declines or degrades as extraction proceeds faces a different sort of problem from that of the firm whose asset in the form of inventory is homogeneous. The firm facing stock-dependent costs must take into consideration the effects of its actions in the current period not only on current period profit, but also on future period costs.

By choosing to extract a unit in the current period, the firm is, in some sense, consuming part of the value of its mineral asset. That reduction in value must be compensated for, if the firm's wealth is to be preserved.

We will show that the quantity which would normally be considered as "surplus" or profit is really no such thing. Rather, it represents the amount of the mineral asset value which is

Figure 2



transformed through extraction and sales into cash, or some other form of asset.

In showing this, we may begin by noting that one of the components of current period net revenue, or  $\phi_t$ , represents the opportunity cost of the units extracted. This is equal to  $\gamma_t \theta_t$ . It is generally recognized that this is not profit in the usual sense, but rather compensation to the firm for the asset value lost when units are extracted in the current period. However, the remainder of net revenue, what we have called the current period "surplus", is not profit either. It also represents compensation to the firm for the reduction in value of the deposit.

This can be shown using the optimality condition for extraction under stock dependent costs, derived from optimizing the relevant current-value Hamiltonian. For the model used in this paper, the relevant expression is

$$\dot{H} = P_t Y_t - C(Y_t, S_t) - \theta_t Y_t$$

From the first order conditions for interior solutions, we have

$$(P_t - \theta_t) = C_Y(Y_t, S_t),$$

$$\theta_t - (1+\delta)\theta_{t-1} = C_S(Y_t, S_t)$$

Rearranging the second expression yields

$$\theta_t - \theta_{t-1} = \delta \theta_{t-1} + C_S(Y_t, S_t)$$

This condition says that the change in asset value is equal to the appreciation of the previous period's value, minus the loss from the induced cost increase (recall that cost is decreasing in stock size).

To show that this reduction in value is exactly offset by the current period "surplus", we will exploit another characteristic of linear homogeneous functions. The relationships between the partial derivatives of the original cost function  $C(Y_t, S_t)$  and the derivative of the modified function  $c(\gamma_t)$  are

$$\frac{\partial C(Y_t, S_t)}{\partial Y_t} = c'(\gamma_t),$$

$$\frac{\partial C(Y_t, S_t)}{\partial S_t} = c(\gamma_t) - \gamma_t \cdot c'(\gamma_t).$$

Substituting this second expression into the condition above yields

$$\theta_t - \theta_{t-1} = \delta \theta_{t-1} + [c(\gamma_t) - \gamma_t \cdot c'(\gamma_t)]$$

If the extraction path is optimal, the other optimality condition must also be satisfied:

$$P_t - \theta_t = c'(\gamma_t^*)$$

Then, along an optimal path, we have

$$\begin{aligned} \theta_t - \theta_{t-1} &= \delta \theta_{t-1} + [c(\gamma_t^*) - (P_t - \theta_t) \gamma_t^*] \\ &= \delta \theta_{t-1} - [(P_t - \theta_t) \gamma_t^* - c(\gamma_t^*)] \end{aligned}$$

The last term is the definition of the current period "surplus".<sup>3</sup>

This indicates that there may be yet another interpretation of the marginal revenue-marginal cost equality condition for optimality. At the point at which price less user cost equals  $c'(\gamma)$ , the loss in future asset value is just equal to the current period gain.

This would not be true at any other value of  $\gamma$ , the proportion to be extracted. Consider first the result of "over-extraction", of extracting a proportion greater than that dictated by the marginal net revenue-marginal cost equality. Should the firm choose to do that, it will degrade the quality of the deposit below the level required to provide the future net benefits computed in the original dynamic programming solution. The increase in current period surplus from doing so will necessarily be less than the loss in future stock value.

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<sup>3</sup> It is a short step from this equation of motion of asset value to the equation defining the movement of user cost shown in the last section. Rearranging the expression above, we have

$$\begin{aligned} (1+\delta)\theta_{t-1} &= (P_t - \theta_t) \gamma_t^* - c(\gamma_t^*) + \theta_t \\ &= P_t \gamma_t^* - c(\gamma_t^*) + (1 - \gamma_t^*) \theta_t \\ &= \phi_t + (1 - \gamma_t^*) \theta_t \end{aligned}$$

Dividing through by  $(1+\delta)$  yields the equation of motion of user cost.



Figure 3 illustrates the loss from over-extraction in a given period. The optimal proportion to extract is  $\gamma^*$ , and the actual extraction level is  $\gamma'$ . The lightly shaded area represents the increase in current period revenue from increasing extraction from  $\gamma^*$  to  $\gamma'$ . That area, plus the heavily-shaded triangle, together represent the loss of asset value, or future profits, from extracting the larger amount in the current period. The darkly-shaded area corresponds to the amount of the reduction in future period value which is not compensated for by current period net revenue. Therefore, it represents pure loss, which cannot be recovered at any point later in the production program.

Now, consider the situation if the firm chooses to extract less than the computed optimal proportion. Here, the firm's loss is due not to the degradation of the deposit, but rather, to the operation of the time constraint resulting from the original assumption concerning the length of the planning horizon.

The user cost ( $\theta_t$ ) at each period is determined by computing the maximum value of stock in all future periods. This maximum value, then, is a function of, inter alia, the number of periods remaining. The firm's opportunities to earn revenue in the future are limited not only by the degradation of the stock, but also by the "degradation" of the assumed economic life of the deposit. Should the firm choose to under-extract in any given period, there will be no future opportunities in which to make up the profit foregone in that period.

Figure 4 illustrates this. Again, the optimal extraction level is  $\gamma^*$ , and the "wrong" level chosen is  $\gamma'$ . The lightly shaded area represents current net revenue foregone which can be made up through extraction in later periods. The heavily shaded area represents current net revenue sacrificed for which there is no later compensation. If not claimed in the current period, this gain is lost forever.

Therefore, it is only at the previously-determined optimum that the firm is certain of collecting a surplus of net revenue over discounted future stock value which exactly offsets the degradation, or decline in the value, of stock from current extraction.

Figure 3  
Effects of Overextraction

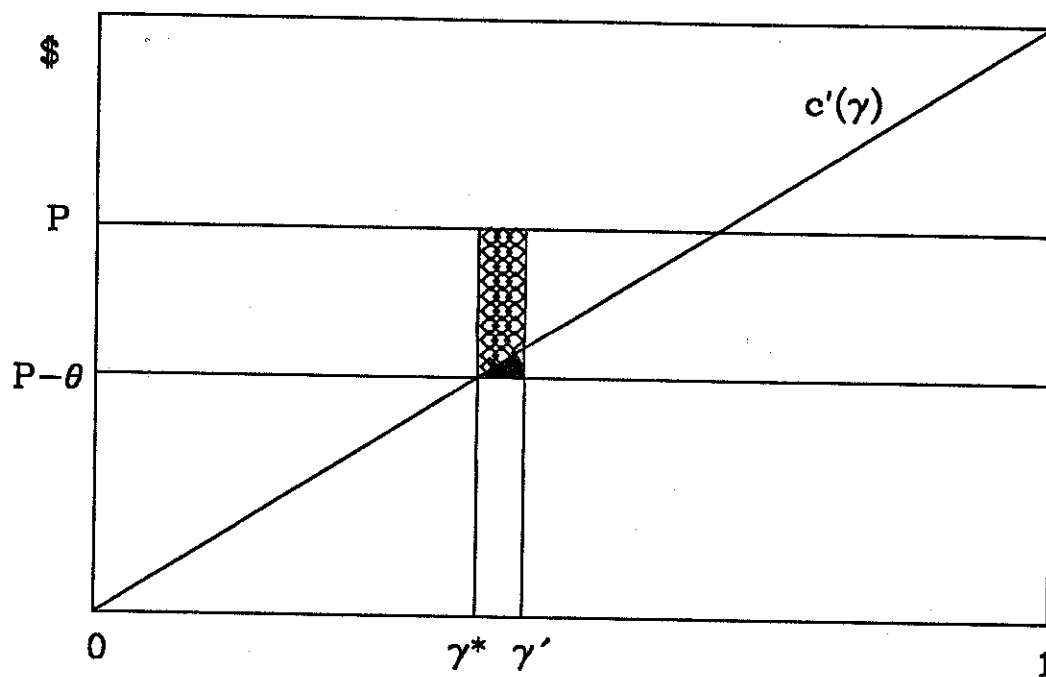
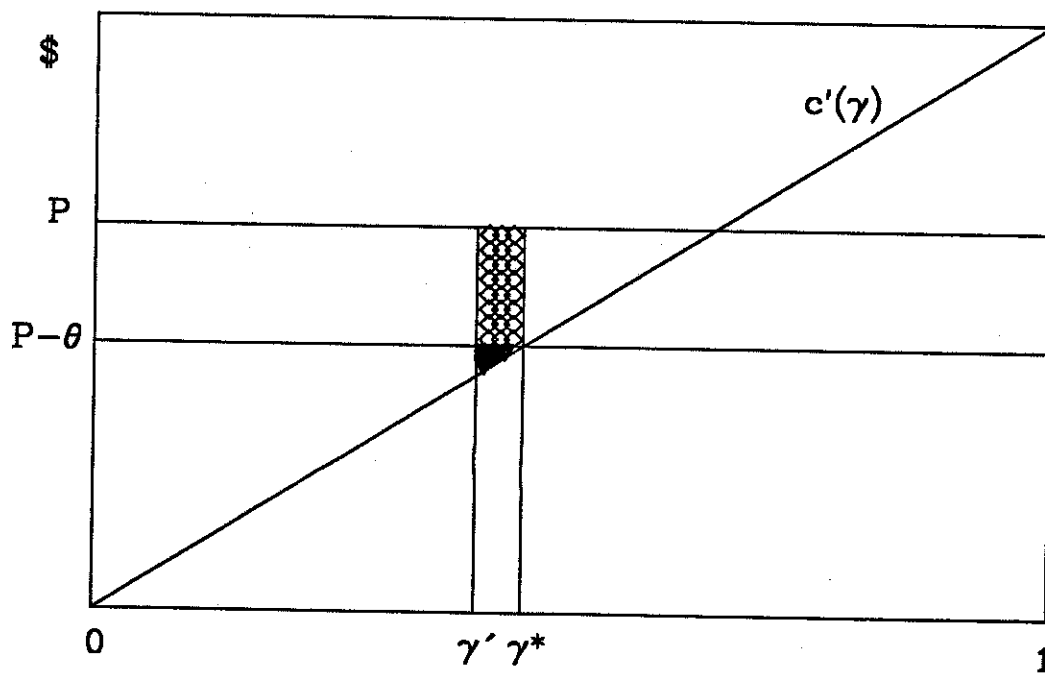


Figure 4  
Effects of Underextraction



In light of the general homage given to Hotelling, it is worth noting the operation of a type of r-per-cent rule in this case.<sup>4</sup> Although price is assumed parametric, ruling out a strict interpretation of the Hotelling rule, we do find the value of the firm's assets increasing at the discount rate, due to changes in production.<sup>5</sup>

This can be seen from the equation describing the movement in user costs. The change in user cost is equal to the original period's user cost multiplied by the discount rate, less the amount drawn off as "surplus". Since this surplus amount represents the portion of the value of the mineral asset which is transformed into an asset of a different type, the value of the firm's total assets, originally derived from its mineral holdings, does grow at exactly the discount rate.

#### COMPARATIVE DYNAMICS OF PRICE CHANGES

We turn now to a discussion of the effects on the optimal extraction profile of one or more changes in the prices received by the firm. These price changes may be brought about by external factors, such as changes in world demand or supply, or through the introduction or modification of production or sales taxes levied on the firm. For our purposes, these are simply different causes of the same type of price change, giving rise to the same effects on production.

For general stock-dependent cost problems, the effects on production of changes in prices can be difficult to untangle. While an increase in  $P_t$  can be expected to result in an increase in production in period  $t$ , if prices in previous periods increase, increased extraction in those periods may drive up period  $t$  costs by so much that production in period  $t$  will actually decline.

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<sup>4</sup> Cairns (1990) suggests that credit for this type of rule really belongs to Gray. Certainly, the effects of discounting future returns are discussed in his "Rent Under the Assumption of Exhaustibility" (1914), and it is but a short step from his numerical example to an r-per-cent rule governing the change in stock value along an optimal production path.

<sup>5</sup> A similar result was demonstrated graphically for the case of U-shaped average costs, which are not stock-dependent, by Scott (1967).

When the cost function is homogeneous of degree one, however, the effects of previous period production changes on current cost can be specified with certainty. In such cases, expressions for the effects of changes in prices on interior<sup>6</sup> values of  $\gamma_t$  can be derived and signed unambiguously. Knowledge of these effects is not the same as knowledge of the changes in each absolute production level, but it provides the means by which the direction of these production changes can be ascertained.

We know that, for any period  $t$ , an interior optimum value of  $\gamma$  ( $\gamma^*$ ), must satisfy the condition

$$P_t - \theta_t = c'(\gamma_t^*) .$$

This condition implies the existence of an optimal value function:

$$\gamma_t^* = \hat{\gamma}_t^*([P_t - \theta_t]) = \gamma_t^*(P_t, \theta_t)$$

Since the function  $c(\gamma_t)$  is increasing and convex in  $\gamma_t$ , the optimal value function  $\gamma^*(P_t, \theta_t)$  is increasing and concave in  $P_t$ , and decreasing in  $\theta_t$ . Therefore,

$$\left( \frac{\partial \hat{\gamma}_t^*([P_t - \theta_t])}{\partial [P_t - \theta_t]} \right) \left( \frac{\partial [P_t - \theta_t]}{\partial P_t} \right) = \frac{\partial \gamma_t^*(P_t, \theta_t)}{\partial P_t} > 0$$

We may also determine a result which will be needed later:

$$\begin{aligned} & \left( \frac{\partial \hat{\gamma}_t^*([P_t - \theta_t])}{\partial [P_t - \theta_t]} \right) \left( \frac{\partial [P_t - \theta_t]}{\partial \theta_t} \right) \\ &= \frac{\partial \gamma_t^*(P_t, \theta_t)}{\partial \theta_t} = - \frac{\partial \gamma_t^*(P_t, \theta_t)}{\partial P_t} < 0 \end{aligned}$$

Because optimal per-unit profit,  $\phi_t$ , is a function of  $P_t$  and  $\gamma_t^*$ , it may also be written as a function of  $P_t$  and  $\theta_t$ :

$$\phi_t = \phi_t(P_t, \theta_t) = P_t \cdot \gamma_t^*(P_t, \theta_t) - c(\gamma_t^*(P_t, \theta_t))$$

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<sup>6</sup> Boundary values of  $\gamma_t$  present special problems. If  $\gamma_t=0$ , we know that  $[P_t - \theta_t] \leq 0$ , but we do not know by how much  $P_t$  will need to increase to change this. Similarly, if  $\gamma_t=1$ , it must be true that  $[P_t - \theta_t] \geq c'(1)$ , but this does not tell us how much  $P_t$  must decrease before  $\gamma_t$  begins to decline.

The change in  $\phi_t$  which is induced by a change in  $P_t$  is:

$$\begin{aligned}\frac{\partial \phi_t(P_t, \theta_t)}{\partial P_t} &= P_t \left( \frac{\partial \gamma_t^*}{\partial P_t} \right) + \gamma_t^* - c'(\gamma_t^*) \left( \frac{\partial \gamma_t^*}{\partial P_t} \right) \\ &= \gamma_t^* + (P_t - c'(\gamma_t^*)) \left( \frac{\partial \gamma_t^*}{\partial P_t} \right) > 0\end{aligned}$$

since, at the optimum,  $P_t$  is greater than  $c'(\gamma_t^*)$  for all  $t < T-1$ , and  $P_{T-1}$  is equal to  $c'(\gamma_{T-1}^*)$

The effects on  $\gamma_t^*$  and  $\phi_t$  of a change in  $P_t$  are shown in Figure 5. The change in  $\gamma_t^*$  is represented by the horizontal distance  $[\gamma - \gamma']$ . The change in  $\phi_t$  is made up of two parts. The heavily shaded area represents the increase in  $\phi_t$  on the old proportion extracted ( $\gamma_t^*$  before the price change). The area shaded more lightly represents the increase in  $\phi_t$  resulting from the induced change in  $\gamma_t^*$ .

Next, we turn to the effects on  $\gamma_t^*$  and  $\phi_t$  of a change in a future period price. Note that while the user cost term in the current period,  $\theta_t$ , is independent of the current period price,  $P_t$ , it does depend on future prices ( $P_{t+1}, P_{t+2}, \dots, P_{T-1}$ ). Then we may write

$$\gamma_t^* = \gamma_t^*(P_t, \theta_t(P_{t+1}, P_{t+2}, \dots, P_{T-1}))$$

The effects on the values of  $\gamma_t^*$  (the optimal proportion to extract) and  $\phi_t$  (current period net revenue) of a change in any future price,  $P_{t+j}$ , are described by the following expressions:

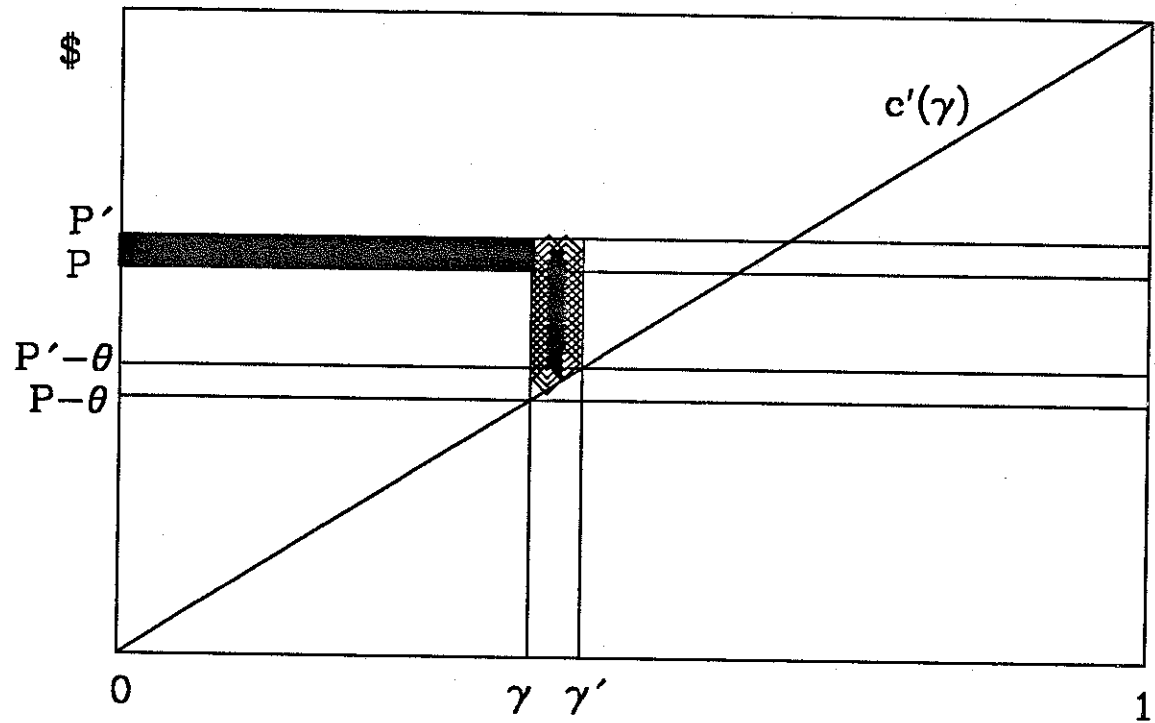
$$\frac{\partial \gamma_t^*(\cdot)}{\partial P_{t+j}} = \left( \frac{\partial \gamma_t^*}{\partial \theta_t} \right) \left( \frac{\partial \theta_t}{\partial P_{t+j}} \right)$$

$$\begin{aligned}\frac{\partial \phi_t(\cdot)}{\partial P_{t+j}} &= \left( \frac{\partial \phi_t}{\partial \gamma_t^*} \right) \left( \frac{\partial \gamma_t^*}{\partial \theta_t} \right) \left( \frac{\partial \theta_t}{\partial P_{t+j}} \right) \\ &= (P_t - c'(\gamma_t^*)) \left( \frac{\partial \gamma_t^*}{\partial \theta_t} \right) \left( \frac{\partial \theta_t}{\partial P_{t+j}} \right)\end{aligned}$$

The signs of some components of these are known, specifically:

$$\frac{\partial \gamma_t^*(\cdot)}{\partial \theta_t} < 0, \quad [P_t - c'(\gamma_t^*)] \geq 0$$

Figure 5  
Effects on  $\gamma$  and  $\Phi$  of  
Changes in Current Period Price



Therefore, the signs of the expressions describing the effects of changes in future prices on current  $\gamma_t^*$  and  $\phi_t$  depend on the sign of the expression describing the effect on the current period user cost ( $\theta_t$ ) of a change in a future price:

$$\frac{\partial \theta_t}{\partial P_{t+j}}$$

Intuition tells us that this is non-negative, since an increase in a future price cannot decrease that future period's profit, and will generally increase the value of units left in situ. We turn now to demonstrating that this is, in fact, the case.

To derive the expression for the effect of a change in any future price on  $\theta_t$ , we begin by writing  $\theta_t$  as a function of all future values of  $\theta$  and  $P$ :

$$\theta_t = \theta_t(P_{t+1}, \theta_{t+1}(P_{t+2}, \theta_{t+2}(P_{t+3}, \dots, \theta_{T-2}(P_{T-1}) \dots))$$

A change in  $P_{t+j}$ , for  $j > 1$ , alters current user cost only indirectly, through its effects on future values of  $\theta$ . Therefore, the effect on  $\theta_t$  of a future price change is composed of a chain of effects, beginning with the direct effect on  $\theta_{t+j-1}$  of the change in  $P_{t+j}$ :

$$\frac{\partial \theta_t}{\partial P_{t+j}} = \left( \frac{\partial \theta_t}{\partial \theta_{t+1}} \right) \left( \frac{\partial \theta_{t+1}}{\partial \theta_{t+2}} \right) \dots \left( \frac{\partial \theta_{t+j-2}}{\partial \theta_{t+j-1}} \right) \left( \frac{\partial \theta_{t+j-1}}{\partial P_{t+j}} \right)$$

To ascertain the sign of this expression, we must first determine the effect on current period user cost of a change in price in the period immediately following. Recall that

$$\begin{aligned} \theta_t &= \rho \{ \phi_{t+1} + (1 - \gamma_{t+1}^*) \theta_{t+1} \} \\ &= \rho \{ \phi_{t+1}(P_{t+1}) + [1 - \gamma_{t+1}^*(P_{t+1})] \theta_{t+1} \} \\ &= \theta_t(P_{t+1}, \theta_{t+1}) \end{aligned}$$

Then the effect of a change in  $P_{t+1}$  on  $\theta_t$  is

$$\begin{aligned}
\frac{\partial \theta_t}{\partial P_{t+1}} &= \rho \left[ \frac{\partial \phi_t}{\partial P_{t+1}} - \theta_{t+1} \frac{\partial \gamma_{t+1}^*}{\partial P_{t+1}} \right] \\
&= \rho \left[ \gamma_{t+1}^* + (P_{t+1} - c'(\gamma_{t+1}^*)) \frac{\partial \gamma_{t+1}^*}{\partial P_{t+1}} - \theta_{t+1} \frac{\partial \gamma_{t+1}^*}{\partial P_{t+1}} \right] \\
&= \rho \left[ \gamma_{t+1}^* + (P_{t+1} - c'(\gamma_{t+1}^*) - \theta_{t+1}) \frac{\partial \gamma_{t+1}^*}{\partial P_{t+1}} \right] \\
\rightarrow (1+\delta) \frac{\partial \theta_t}{\partial P_{t+1}} &= \gamma_{t+1}^* + (P_{t+1} - c'(\gamma_{t+1}^*) - \theta_{t+1}) \frac{\partial \gamma_{t+1}^*}{\partial P_{t+1}} \quad = \gamma_{t+1}^* > 0
\end{aligned}$$

The expression to the right of the equals sign in the last row is represented by the shaded areas in Figure 6. The lightly-shaded rectangle is the gain in  $(1+\delta)\theta_t$  which results from an increase in  $P_{t+1}$  on the original optimal proportion extracted. The small, darkly-shaded triangle represents a small additional gain in  $(1+\delta)\theta_t$  due to the change in  $\gamma_{t+1}$ . (For small changes in price, this second term may be ignored.)

(We may also state a result which will be used later. This is that the change in stock value  $((1+\delta)d\theta_t)$  must be less than the change in price  $(dP_{t+1})$ , if  $\gamma_{t+1}^*$  is interior (lying within the interval  $(0,1)$ ) both before and after the change.)

The effect on  $\theta_t$  of a change in  $\theta_{t+1}$  (which is, in turn, a function of  $P_{t+2}$ ,  $\theta_{t+2}$ , and future values of these) is:

$$\begin{aligned}
\frac{\partial \theta_t}{\partial \theta_{t+1}} &= \rho \left[ \left( \frac{\partial \phi_{t+1}}{\partial \gamma_{t+1}^*} \right) \left( \frac{\partial \gamma_{t+1}^*}{\partial \theta_{t+1}} \right) - \theta_{t+1} \left( \frac{\partial \gamma_{t+1}^*}{\partial \theta_{t+1}} \right) + (1 - \gamma_{t+1}^*) \right] \\
&= \rho \left[ (P_{t+1} - c'(\gamma_{t+1}^*) - \theta_{t+1}) \left( \frac{\partial \gamma_{t+1}^*}{\partial \theta_{t+1}} \right) + (1 - \gamma_{t+1}^*) \right] \\
\rightarrow (1+\delta) \frac{\partial \theta_t}{\partial \theta_{t+1}} &= \left\{ (P_{t+1} - c'(\gamma_{t+1}^*) - \theta_{t+1}) \left( \frac{\partial \gamma_{t+1}^*}{\partial \theta_{t+1}} \right) + (1 - \gamma_{t+1}^*) \right\} \\
&= (1 - \gamma_{t+1}^*) > 0
\end{aligned}$$

The expression in braces is shown as the shaded areas in Figure 7. The entire shaded rectangle represents the gain in  $(1+\delta)\theta_t$  which results from the increase in  $\theta_{t+1}$  on that proportion which was not being extracted before the change. The small,



Figure 6  
Effect on  $(1+\delta)\theta_{t-1}$   
Changes in Current Period Price ( $P_t$ )

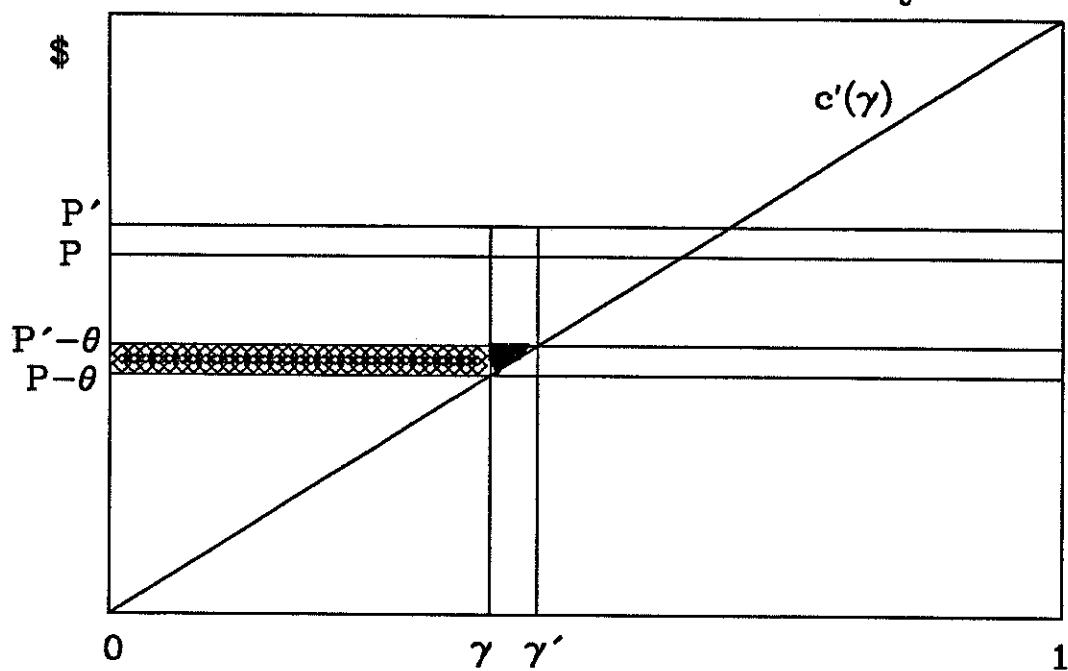
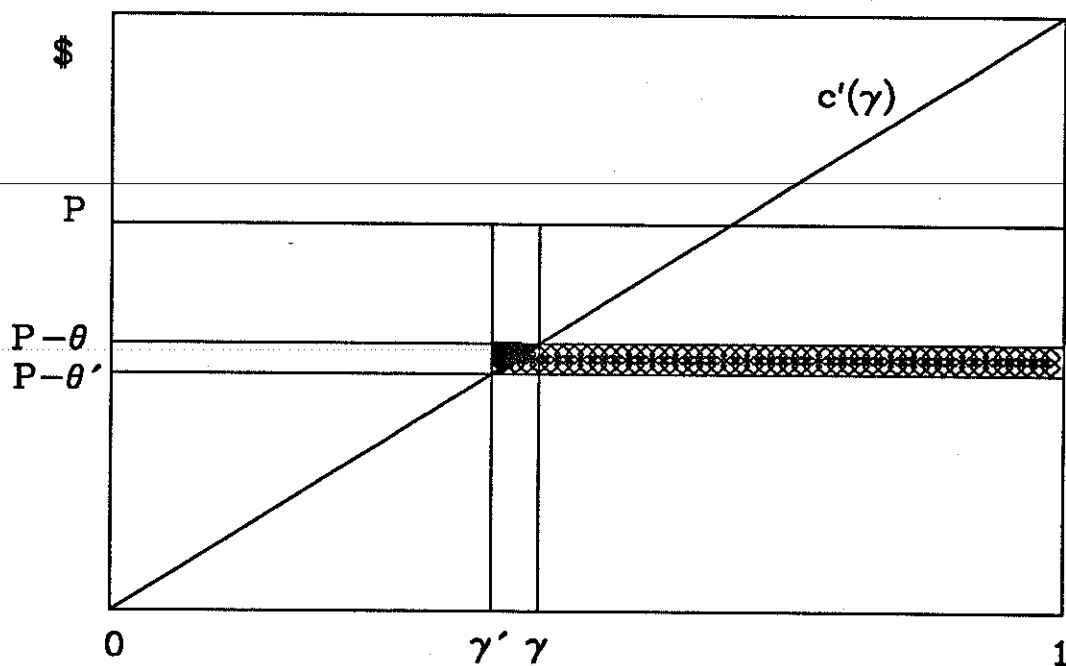


Figure 7  
Effect on  $(1+\delta)\theta_t$  of  
an Increase in  $\theta_{t+1}$



darkly-shaded triangle represents the adjustment to this gain from the induced decrease in  $\gamma_{t+1}^*$ . (Again, for small changes, this may be ignored.)

Because every component of the product defining it is strictly positive, we are now able to conclude that

$$\frac{\partial \theta_t}{\partial P_{t+j}} > 0$$

and, therefore,

$$\frac{\partial \gamma_t^*(\cdot)}{\partial P_{t+j}} < 0, \quad \frac{\partial \phi_t}{\partial P_{t+j}} < 0.$$

With these expressions, we may now describe the effects of temporary or permanent price changes on the entire extraction path.

When discussing the effects on the extraction path of single- or multi-period changes in prices, a distinction must be drawn between situations in which future price changes are anticipated in advance and situations in which price changes are known only in the period(s) in which they occur. While the second circumstance would likely take place more frequently, there are occasions, such as announced tax changes, when a future price change may be known before it takes effect.

We examine first the situation in which the price changes in a single period, period  $\tau$ . Without loss of generality, for explanatory purposes we will consider the effects of a price increase. Throughout this discussion, we continue to assume that  $\gamma_t^*$ , the optimal proportion to extract in each period, lies strictly between zero and one.

Little needs to be added to the expressions so far derived when the price change is known only when it occurs. An increase in  $P_\tau$  will result in increases in  $\gamma_\tau$  and  $\phi_\tau$ . The values of  $\gamma_t, \phi_t$ , and  $\theta_t$  for all future periods ( $t > \tau$ ) will remain unchanged. It should be noted, however, that while the optimal proportion of reserves to extract will remain constant for those future periods, the absolute size of the reserve available in each period will have been reduced by the increase in quantity extracted in the period in which the price increased. Therefore, output in the current period will increase, and output in all future periods will be reduced proportionally.

When the single-period price increase is known one or more periods before it occurs, while later ( $t > \tau$ ) values of  $\gamma_t$  will remain unchanged, the optimal proportions to extract in earlier

periods ( $t < \tau$ ) will be reduced. This is because an increase in  $P_t$  induces an increase in  $\theta_{t-1}$ , which, in turn, induces a (smaller) increase in  $\theta_{t-2}$ , and so on. These increases in user costs result in smaller amounts being extracted in earlier periods. Intuitively, the firm is simply holding onto larger supplies in order to take advantage of the increased price in the future.

Next we examine the case in which the same price change occurs in all periods from  $\tau < T$  to the end of the horizon ( $dP_t = dP$  for all  $\tau \leq t < T$ ). For each period  $t \geq \tau$  except the last (in which the user cost term is zero), there are two effects on the proportion to extract, which are of opposite sign. The increase in price in the current period will act to increase  $\gamma_t^*$ , the optimal proportion to extract. However, the increase in future period prices raises the user cost term ( $\theta_t$ ), which acts to decrease  $\gamma_t^*$ .

We will show that the effect of the price increase outweighs the effect of the increase in user cost, so the net effect will be an increase in the optimal proportion to extract in every period in which the price increases.

To prove this, we must show that

$$\begin{aligned} d[P_t - \theta_t(\cdot)] &= dP - d\theta_t(\cdot) > 0 \\ \rightarrow dP &> d\theta_t(\cdot) \end{aligned}$$

Rather than showing this directly, we will prove that, for all  $t \geq \tau$ :

$$dP > (1+\delta)d\theta_t \geq d\theta_t$$

The total differential of  $\theta_t$  is

$$\begin{aligned} d\theta_t &= \frac{\partial \theta_t}{\partial P_{t+1}} dP + \frac{\partial \theta_t}{\partial \theta_{t+1}} d\theta_{t+1} \\ &= \rho \left\{ \gamma_{t+1}^* + [P_{t+1} - C'(\gamma_{t+1}^*) - \theta_{t+1}] \frac{\partial \gamma_{t+1}^*}{\partial P_{t+1}} \right\} dP \\ &\quad + \rho \left\{ (1 - \gamma_{t+1}^*) + [P_{t+1} - C'(\gamma_{t+1}^*) - \theta_{t+1}] \frac{\partial \gamma_{t+1}^*}{\partial \theta_{t+1}} \right\} d\theta_{t+1} \end{aligned}$$

Using the relation

$$\frac{\partial \gamma_t^*}{\partial \theta_t} = - \frac{\partial \gamma_t^*}{\partial P_t}$$

this may be shown to imply

$$\begin{aligned}
(1+\delta)d\theta_t &= \gamma_{t+1}^* dP + [P_{t+1} - c'(\gamma_{t+1}^*) - \theta_{t+1}] \left( \frac{\partial \gamma_{t+1}^*}{\partial P_{t+1}} \right) dP \\
&\quad + d\theta_{t+1} - \gamma_{t+1}^* d\theta_{t+1} - [P_{t+1} - c'(\gamma_{t+1}^*) - \theta_{t+1}] \left( \frac{\partial \gamma_{t+1}^*}{\partial P_{t+1}} \right) d\theta_{t+1} \\
&= d\theta_{t+1} + \left\{ \gamma_{t+1}^* + [P_{t+1} - c'(\gamma_{t+1}^*) - \theta_{t+1}] \left( \frac{\partial \gamma_{t+1}^*}{\partial P_{t+1}} \right) \right\} (dP - d\theta_{t+1})
\end{aligned}$$

The last expression in braces is equal to

$$\frac{\partial \theta_t}{\partial P_{t+1}} > 0$$

which is shown in Figure 6. If  $\gamma_t^*$  lies within the interval (0,1), this expression must be strictly less than one.

We may now write

$$(1+\delta)d\theta_t = d\theta_{t+1} + \left\{ \frac{\partial \theta_t}{\partial P_{t+1}} [dP - d\theta_{t+1}] \right\}$$

Further,

$$\begin{aligned}
dP > d\theta_{t+1} &\rightarrow dP > \left[ d\theta_{t+1} + \left\{ \frac{\partial \theta_t}{\partial P_{t+1}} [dP - d\theta_{t+1}] \right\} \right] \\
&\rightarrow dP > (1+\delta)d\theta_t \geq d\theta_t
\end{aligned}$$

Finally, we note that  $\theta_{T-1}$ , the user cost term in the final period, is zero. Then  $dP > \theta_{T-1}$ , implying that  $dP > d\theta_t$  for all  $t \geq \tau$ .

If the "permanent" price change becomes known only in period  $\tau$ , the analysis is complete. Any changes to quantities in periods before  $\tau$  are irrelevant, since the firm cannot alter past activities. However, if the change can somehow be known some time before it occurs, production in those periods between the present and period  $\tau$  will decrease. This is due to the effects of increased user costs ( $\theta$ 's) which are not offset by increases in price in those periods.

The effects of price changes on absolute quantities extracted may now be ascertained. In all time periods in which a price increase has occurred, the optimal proportion of reserves to extract increases. This results in higher absolute production in the first of these periods (period  $\tau$ ), implying that a smaller

proportion of reserves is passed on to the next period. In the case of a "permanent" price change, there will be a greater proportion extracted in the next period as well, resulting in an even smaller amount of reserves being left for period  $\tau+2$ . If the span of time defined by  $T-\tau$  is long enough, the increases in extraction rates in those periods immediately following period  $\tau$  can so deplete remaining reserves that absolute quantities extracted in the final production periods actually decrease.

When a multi-period price increase occurs in a model such as this, it may appear that production has been shifted forward. Superficially, the result may, at first glance, resemble that described by Gray (1914) more than 75 years ago. In fact, such an interpretation is misleading. In Gray's analysis, the entire deposit is exhausted, whereas in this model, generally there will be some units which are abandoned due to the high cost of extracting them. Also, in Gray's work, production is shifted forward due to the existence of a positive discount rate. In this model, we find early period production increasing even when the discount rate is zero.

The correct interpretation takes into account not only the increase in production in the early periods, but also the decline in the amount abandoned. The rise in price induces the firm to extract more from the deposit, and it chooses to divide that increase intertemporally in order to best take advantage of its profit opportunities. Since costs are lower in the early part of the production horizon, the largest increases in extraction are observed in these periods.

Increases in early period production, then, do not necessarily imply that later period production falls. We observe early production rising and later production falling only when the price increase is relatively large. In such a case, the increases in proportions extracted in early periods are so large that reserves in later periods are significantly diminished.

In light of the fact that the quantity of original stock which is ultimately abandoned is a function of output price, it may be necessary to specify more precisely the definition of "reserves". In the model used here, the original reserve,  $S_0$ , is implicitly defined in physical, rather than economic terms. The proportion of  $S_0$  which can be economically extracted over the specified time interval depends on the output price. If reserves are defined in an economic sense, as, for example, "the known amounts of a mineral that can be profitably produced at current prices using current technology"<sup>7</sup>, we find that reserves themselves are endogenous in this model. A price increase changes not only the pattern of

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<sup>7</sup> Fisher (1981), pp. 94.

production, but also the amount of the physical stock of the mineral which will ultimately be recovered.

This is a significant result for the formulation of minerals policy. The imposition of a severance tax (a specific tax on the amount removed) or an ad valorem tax on the sale of the mineral, will decrease the amount of the reserve which a firm will find it profitable to extract, thereby effectively shrinking the total supply.

## CONCLUSIONS

The model described in this paper is an artificial one, amenable to being solved using a relatively simple technique only because of the restrictions imposed on both cost (linear homogeneity in output and remaining stock) and revenue (price-taking behavior). However, this model is arguably less simple than models in which marginal costs are assumed constant. The imposition of restrictions, by removing a number of possible empirical complications, permits us to focus more clearly on the basic characteristics of extraction problems in which costs increase with depletion.

There are two major aspects of non-renewable resource exploitation which are brought out by the model. First, it shows that the firm chooses to extract and thereby degrade the quality of stock only insofar as its current period net revenue suffices to compensate for the loss in mineral asset value incurred by doing so. The firm effectively chooses a pattern of extraction which preserves the maximum value of the mineral reserve at each point in time.

Second, the model illustrates the difference between a physical definition of reserves, under which the ultimate reserve size is fixed, and therefore exogenous, and an economic definition, under which the quantity of reserves is really endogenous, because it depends on the economic conditions facing the firm.

In addition, the fact that the optimal solution lends itself easily to a graphical analysis means that this model can be of use as a pedagogical device. The generic diagram illustrating the optimal production in any period  $t$  depicts all four quantities of interest to an analyst. It shows the optimal proportion to extract, the net revenue to be earned by doing so, the internal value of reserves to the firm, and the user cost.

None of these conclusions can be derived from a model of extraction under stock-dependent costs in which the cost function is completely unrestricted. As international trade economists have long been aware, models which are artificially simple can sometimes focus our attention on results which get buried in models which are more realistic, and hence more complicated.

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