CORNELL AGRICULTURAL ECONOMICS STAFF PAPER

A NOTE ON THE OPTIMAL HEDGE RATIO: A REVISIT

by

Donald J. Liu

and

Terry L. Roe

April 1986

Staff Paper 86-17

Department of Agricultural Economics
Cornell University Agricultural Experiment Station
New York State College of Agriculture and Life Sciences
A Statutory College of the State University
Cornell University, Ithaca, New York, 14853

A Note on the Optimal Hedge Ratio: A Revisit

Based on the empirical evidence, Brown (1985) argued that, in order for a portfolio approach to hedging to be useful, one must be able to empirically demonstrate that hedge ratios are less than one. After a "reformulation" of the conventional portfolio model [e.g. Markowitz (1959)], he was able to show that the resulting hedge ratios for some selected storable commodities are not significantly different from one. Hence, he concluded that, unless some supportive evidence can be found elsewhere, the portfolio approach to hedging should be questioned. Brown defended his "reformulation" of the conventional portfolio model on the grounds that it yields a model which is both theoretically and statistically superior.

The first purpose of this paper is to point out the flaw of Brown's hypothesis and, hence, the conclusion drawn. An optimal hedge ratio derived from a portfolio model can be either greater or less than unity; examining the size of its empirical counterpart cannot be used to either support or reject the underlying portfolio model. Rather, the debate is focused on the empirical appropriateness of the storage cost theory advanced by Working (1949). Based on the theory, it will be argued that the optimal hedge ratio tends to be less than one for storable commodities. Thus, if Brown's measure of the optimal hedge ratio is indeed superior, his results would imply the rejection of the storage cost theory.

It will be demonstrated that Brown's "reformulation" of the conventional portfolio model is nothing more than a redefinition of a hedge ratio; one which is not appropriate. Moreover, because either a discount or a premium relationship between cash and futures prices can exist (e.g. Tomek and Robinson, p236; 1981), Brown's hedge ratio will be shown to bear no restriction as to the magnitude of its deviation from one. Hence, examining the size of the corresponding estimated hedge ratio cannot be used as a basis for rejecting the storage cost theory mentioned.

A second somewhat different, but related, purpose of this paper is to extend the conventional two-period portfolio model to a dynamic one so as to illustrate the need for a more general treatment of a hedge ratio in future research. Specifically, the traditional static approach of regressing the cash price on the futures price to determine the optimal hedge ratio may be inappropriate. In the example given, the movement of futures price over time is shown to also play a major role in determining the optimal hedge.

A Static Portfolio Model

Consider a two-period model where at time 1 the agent has to make a cash commitment for time 2. Also, the agent can take a position on the futures market with the contract maturing at time 2 (contract 2 hereafter). Denote the cash and futures positions as Q and $H_{1,2}$, respectively, where a long position will be reflected by a positive value and a short position by a negative value. Also, denote time t (t=1,2) cash and futures prices as P_t and $F_{t,2}$, respectively.[1]

Following Berck (1981), the agent chooses the optimal cash and futures positions by maximizing the expected profit in the next time period adjusted for risk, where risk is measured by the variance of profit.

$$(1)\quad \max_{\{Q,\ H_{1,2}\}} \quad Exp[\pi] - \lambda\ VAR[\pi]$$

s.t.

$$(2)\quad E[\pi] = (E[P_2] - P_1)\ Q + (E[F_{2,2}] - F_{1,2})\ H_{1,2}$$

$$(3)\quad Var[\pi] = Q^2\ Var[P_2] + H_{1,2}^2\ Var[F_{2,2}] + 2\ Q\ H_{1,2}\ Cov[P_2, F_{2,2}]$$

where λ is the risk parameter and E, Var, Cov are expectation, variance, and covariance operators, respectively.

For analytical purposes, denote the optimal cash position as Q^*.[2] Then, the optimal futures position becomes:

$$(4)\quad H_{1,2}^* = \frac{E[F_{2,2}] - F_{1,2}}{2\lambda\ Var[F_{2,2}]} - \frac{Cov[P_2, F_{2,2}]}{Var[F_{2,2}]}\ Q^*$$

The first term of the right-hand-side of (4) has nothing to do with the position taken in the cash market. It depends only on the agent's attitudes toward risk and the expected change, as well as variance of the futures price. **It represents speculation**. On the other hand, the second term of the optimal condition depends on the position taken in the cash market, Q^*. It also depends on the ratio of the covariance between next period's cash and futures prices and the variance of the latter. Denote this ratio as

---

1 Consider the case where the agent is a producer. At time 1, the agent has to make a cash commitment as to how much to sell at time 2. Then, P_1 represents the average cost of producing Q and P_2 denotes output price at time 2.

2 Conditions for the separability of cash and futures decisions have been discussed by various authors (e.g. Holthausen, 1979; Chavas and Pope, 1982). In the present case, where basis risk is admitted, cash and futures positions have to be solved simultaneously.

$$(5)\quad \beta^* = \frac{Cov[P_2, F_{2,2}]}{Var[F_{2,2}]} .$$

Because β^* coincides with the regression coefficient of the cash price on the futures price, it measures the relative movement of the two prices in question. Hence, **the second term of (4) represents a hedging component** and it enables the agent to lock in the random net proceeds (excluding the speculative component just mentioned) at the current expected level.

For most practical purposes, the speculative component of (4) can be ignored as there is no underlying quantity to magnify the size of the expected price change over time (like Q^* magnifies β^*). Then, the hedge ratio, which is defined as **the negative ratio of units of hedged stocks to units of total stocks**, under optimality is β^*.[3]

Since futures price and cash price are positively correlated in general, β^* should have a positive sign. In the context of regression analysis, the coefficient bears no restriction as to the direction of its deviation from one. However, if the futures price is the expected spot price and the difference between the two prices at a given time is a reflection of storage costs, then the price change over time for the cash commodity would tend to be larger than changes in the price for the futures contract.[4] **Hence, based on the theory of storage cost, one can argue that β^* should be less than one for storable commodities.**

Brown's Reformulation

Based on the empirical evidence, Brown argued that the portfolio approach to hedging is valid only if hedge ratios can be empirically verified to be less than one. Before proceeding any further, it should be clear that Brown's argument was philosophically wrong. As indicated in (5), the magnitude of an estimated hedge ratio can not be used to either reject or support the underlying portfolio model. With an estimated β^* not

3 Then, this optimal hedge ratio coincides with the minimum risk hedging ratio adopted by Brown (see Kahl, 1983).

4 Heuristically, one can see this by picturing that the cash price "travels" (stochastically) toward the rather stable (but stochastic) futures price over time, according to the cost of storage. This is valid, of course, only for storable commodities; ones to which Brown's analysis applied. Note that inventory adjustment by competitive market participants will keep the difference between cash and futures prices in line with storage costs and, hence, rule out the scenario that the modification of an unwarranted initial expectation (prompted by the available of new information) will force the futures price to "travel" toward the cash price.

less than unity, one can, at best, argue that the price change over time for the cash commodity is not larger than that for the futures contract; a result which casts doubt on the theory of storage cost. Thus, **the debate should rather focus on the empirical appropriateness of the storage cost theory.**

Using (5), Brown estimated β^* for wheat, corn, and soybeans traded on the Chicago Board of Trade. The estimated coefficients were significantly less than one; a result which is consistent with the development so far in the paper. However, Brown argued that, since residuals of price level regression often exhibit significant autocorrelation, the estimated hedge ratios obtained from (5) are statistically inefficient. Moreover, since the object of a hedge in futures markets is to reduce the risk of price changes, he argued that an empirical analysis based on examining the relationship between levels (see (5)) leads to misspecified hedge ratios. Hence, he proposed that a model based on percentage change in price from one time period to the next is both theoretically and statistically superior to that based on price levels. After reformulation, he was able to show that the resulting hedge ratios for the commodities under study were not significantly different from one.

If Brown's reformulation is indeed superior, then his results can be used as evidence against storage cost theory. Should one reject the theory of storage cost?

Brown's argument regarding the problem of serial correlation in a regression analysis is well taken. By specifying a simple regression of the cash price on the futures price, the error term must inevitably be regarded as capturing the effects of the omitted variables. As pointed out in Harvey (1981, p207), aggregation of a number of variables generally leads to an autoregressive-moving averaging process. Hence, the residuals of the regression can exhibit autocorrelation; a violation of the assumptions of the ordinary least squares used to estimate the hedge ratios. Since higher-order serial correlation can also pose a problem, however, there is no guarantee that a regression based on price changes, as proposed by Brown, will yield more efficient estimates of the hedge ratio. Brown only reported Durbin-Watson statistics and there is no way of telling whether high order serial correlations exist in his reformulated models. Note that first order serial correlations were detected in both the conventional and the reformulated models.[5]

Furthermore, even if Brown's reformulation were to result in a more efficient estimate of hedge ratio, it can be shown that no restriction as to the magnitude of his hedge ratio can be derived from the reformulated

5 More fundmentally, Brown's argument about an empirical analysis based on the relationship between levels leading to misspecified hedge ratios was wrong. Since cash and futures prices do not move in a perfect tandem, basis risk is the uncertainty that a hedger has to face. Hence, an optimizing agent should concentrate on the relationship between the two price levels, rather than changes.

model. Hence, one cannot use his results to reject the storage cost theory mentioned.

Specifically, according to Brown, the objective is to maximize the expected return on a portfolio comprised of spot and futures positions adjusted for risk, and this is accomplished by choosing the optimal total values of the spot and futures positions at time 1 prices, P_1Q and $F_{1,2}H_{1,2}$. Since P_1 and $F_{1,2}$ are nonstochastic (i.e. known), it is clear that Brown's reformulation is equivalent to the conventional model of (1) to (3). The only difference is his redefining a hedge ratio as the ratio of the total value of futures position to the total value of the spot position; evaluated at time 1 prices (i.e. $F_{1,2}H_{1,2}/P_1Q$). Denote his optimal hedge ratio as B^*. Then,

$$(6)\quad B^* = \frac{F_{1,2}}{P_1}\beta^*$$

Two questions arise! First, does Brown's definition of a hedge ratio make economic sense? More importantly, given the restriction on the magnitude of β^* for storable commodities, is there anything that one can say about the magnitude of B^*?

Since β^* is the ratio of the optimal units of futures to cash positions, it is a summary statistic derived under optimality. It is a yardstick for efficiency. On the other hand, Brown's hedge ratio B^* does not possess this property as it is biased by a factor which reflects only market conditions of the decision period. Furthermore, since both a premium and a discount relationship between futures and cash prices are plausible, the multiplying factor (i.e. $F_{1,2}/P_1$) in (6) can be greater or less than one. Hence, the magnitude of B^* can not be determined *a priori*. In fact, as indicated by the inflated B^* reported in Brown, one can conclude that premium relationships prevailed in his study period.

A Dynamic Portfolio Model

The model discussed so far is static in nature; it does not admit plan revision prompted by the receipt of new information occuring in the interval of time 1 to time 2. A dynamic portfolio model can provide more insight into the real world day-to-day decision making process. Hence, it can generate a more realistic measure of the optimal hedge ratio and provide a more useful policy prescription.

Consider a three-period model. To preserve the usefulness of previous notations, assume the decision period and the terminal period is time 0 and time 2, respectively (instead of time 1 and time 3). At the initial time the agent is to hold stocks and the stocks has to be sold at time 2. The agent has the option of taking a position on futures market with contract maturing at time 2. At time 1, the agent is assumed to update his information and, hence, modify the contract position. Finally, as time progresses to period 2, all the stocks have to be sold and the futures position cancelled.

Hence, at time 0 the agent has to choose the initial holding of stocks as well as the position of the futures contract. Also, the agent establishes a contingency plan for time 1 position on the contract. The mean-variance objective is:[6]

$$(7)\quad E[\pi] = (E[P_2] - P_0)\, Q + (E[F_{1,2}] - F_{0,2})\, H_{0,2} + (E[F_{2,2}] - E[F_{1,2}])\, H_{1,2}$$

$$(8)\quad Var[\pi] = Q^2\, Var[P_2] + (H_{0,2} - H_{1,2})^2\, Var[F_{1,2}] + H_{1,2}{}^2\, Var[F_{2,2}] + 2\, Q\, (H_{0,2} - H_{1,2})\, Cov[P_2,F_{1,2}] + 2\, Q\, H_{1,2}\, Cov[P_2,F_{2,2}] + 2\, (H_{0,2} - H_{1,2})\, H_{1,2}\, Cov[F_{1,2},F_{2,2}]$$

where P_t, $H_{t,2}$ and $F_{t,2}$ is time t cash price, time t futures position, and the corresponding futures price, respectively.

Insight into the above optimization problem can be analyzed by a dynamic programming backward substitution scheme (e.g. Bertsekas, 1976). Specifically, a policy rule for the time 1 choice variable (i.e. $H_{1,2}$) can first be solved as a function of time 0 choice variables. This is achieved by maximizing (1) with profit and variance of profit being defined in (2) and (3), respectively. Hence, the resulting policy rule is (4). As discussed, the two components in (4) represent time 1 pure speculation on contract 2 and time 1 pure hedging against price risk pertaining to cash market. Denote these two components as $PS_{1,2}$ and $PH_{1,2}[Q^*]$, respectively; where $[Q^*]$ denotes that $PH_{1,2}$ is a function of the optimal cash position. That is,

$$(9)\quad H_{1,2}{}^* = PS_{1,2} + PH_{1,2}[Q^*]$$

Now, differentiate the overall objective (see (7) and (8)) with respect to $H_{0,2}$, set the resulting expressions equal to zero, and substitute the right-hand-side of (9) for $H_{1,2}{}^*$. The optimal time 0 futures position can then be expressed as:

6 For illustration purposes, the model is deliberately simplified. Anderson and Danthine (1981) considered the case where both the discount rate and production uncertainty are incorporated. A model of overlapping production cycles with both input and output price uncertainties can be found in Liu (1985).

$$(10)\quad H_{0,2}^{*} = \frac{E[F_{1,2}] - F_{0,2}}{2\lambda\, Var[F_{1,2}]} - \frac{Cov[P_2,F_{1,2}]}{Var[F_{1,2}]} Q^{*} - \left(\frac{Cov[F_{1,2},F_{2,2}] - Var[F_{1,2}]}{Var[F_{1,2}]}\right) (PS_{1,2} + PH_{1,2}[Q^{*}])$$

Interpretations for the first two terms of the above can be obtained using the same analogy employed for (4). They represent the agent's time 0 speculation and hedging components, respectively. Specifically, they reflect the agent's taking advantage of price changes between time 0 and time 1 in speculation and in risk management. The last two terms entail further discussion. At time 0 the agent anticipates taking a speculative position of $PS_{1,2}$ and a hedging position of $PH_{1,2}$ at time 1. The decisions regarding these two positions will partially be based on the **to be realized** futures price at the time (i.e. $F_{1,2}$ in (4)). Hence, at time 0 the agent is facing an additional random variable $F_{1,2}$; to cope with this uncertainty the agent takes a position at the present time. Thus, the third term of (10) can be interpreted as **a hedge against the anticipated speculation**, while the last term can be interpreted as **a hedge against the anticipated hedging**.

As before, ignore speculation. Then, only the second and the fourth terms of (10) remain; a hedging component and a hedge against the anticipated hedging component. Both terms are a function of the optimal cash holding. Even with this deliberately simplified model, the optimal hedge ratio can no longer be obtained by simply regressing cash price on futures price. As indicated by the last term of (10), the movement of futures price over time also plays a major role in determining the optimal hedge ratio. Note that a hedge against the anticipated futures activity can be regarded as a form of cross hedge (over time). In general, both a cross hedge over time and a cross hedge over commodities can result from a dynamic portfolio model. Thus, the formula for an optimal hedge ratio becomes rather complicated and its magnitude cannot be determined *a priori* even for storable commodities. It should be evaluated with a clear underlying decision model in mind.

Conclusions

This paper presented evidence refuting Brown's contention that a portfolio approach to hedging is not appropriate. Brown's procedure was demonstrated to be invalid because his hypothesis was philosophically wrong and his reformulated model bore no restriction as to the magnitude of the resulting hedge ratio. This paper also extended the conventional static model to a dynamic one so as to illustrate the complexity of an optimal hedge ratio under a more realistic setting. It was shown that the optimal hedge ratio cannot be obtained just by the traditional method of regressing cash price on futures price; a more detail understanding of the underlying economic decision structure is entailed.

References

Anderson, R., and J-P Danthine. *The Time Pattern of Hedging and the Volatility of Futures Prices.* Working Paper Series # CSFM-7, Columbia Business School, April, 1981.

Berck, P. "Portfolio Theory and the Demand for Futures: The Case of California Cotton." *Amer. J. Agr. Econ.* 63(1981): 466-74.

Bertsekas, D. *Dynamic Programming and Stochastic Control.* New York: Academic Press, Inc., 1976.

Brown, S. "A Reformulation of the Portfolio Model of Hedging." *Amer. J. Agr. Econ.* 67(1985): 508-12.

Chavas, J-P, and R. Pope. "Hedging and Production Decisions under a Linear Mean-Variance Preference Function." *Western J. Agr. Econ.* 7(1982): 99-110.

Harvey, A. C. *The Econometric Analysis of Time Series.* New York: John Wiley & Sons, 1981.

Holthausen, D. "Hedging and the Competitive Firm under Price Uncertainty." *Amer. Econ. Review.* 69(1979): 989-95.

Kahl, K. "Determination of the Recommended Hedging Ratio." *Amer. J. Agri. Econ.* 65(1983):603-05.

Liu, D. "The Willingness to Pay of an Individual Fed Cattle Producer for a Better Forecasting Technique: An Evaluation in a Dynamic Economic Setting with Cash and Futures Marketing Alternatives." Ph.D. thesis, University of Minnesota, St. Paul, 1985.

Markowitz, H. *Portfolio Selection: Efficient Diversification of Investment.* New York: John Wiley & Sons, 1959.

Tomek, W. and K. Robinson. *Agricultural Product Prices.* New York: Cornell University, 1981.

Working, H. "The Theory of the Price of Storage." *Amer. Econ. Rev.*, 39(1949): 1254-62.