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WORKING ON FISHER'S PROBLEM WITH COMPUTER ALGEBRA

by Duane Chapman

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Department of Agricultural Economics
Cornell University Agricultural Experiment Station
New York State College of Agriculture and Life Sciences
A Statutory College of the State University
Cornell University, Ithaca, New York, 14853

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Abstract

Application of optimal control theory to applied resource problems has been limited by the difficulty of numerical solutions. Typically, terminal values for the length of the production period, price, or production have been assumed rather than optimized. The use of an objective functional with explicit discounting gives direct solution values for n, y(t), p(t), and monopoly profit (or consumer surplus) for continuous or discrete problems. The method can be used for numerical solutions to problems with linear demand, cost trend, or risk of expropriation. Computer mathematics is a useful tool in exploring solution values for specific parameters. The techniques are illustrated with Fisher's widely used discrete problem, and with application to parameters representing remaining world oil resources for competitive and monopolistic assumptions.

Working on Fisher's Problem with Computer Algebra

The theory of natural resource use over time has undergone considerable evolution since Hotelling's pathbreaking work 50 years ago. Clark, Dasgupta, Gilbert, Fisher, Kamien and Schwartz, Heal, Smith, and Solow have been among the leaders in this effort. However, applications of this theory to numerical problems has been difficult. My purpose here is to describe in a precise, but accessible, way some problems and numerical solutions in oil use, which can be used in teaching theory and analysis. The ambition, then, is small, intending to restate in simple form material which has been thoroughly discussed in the literature.

Some interesting implications arise from the numerical solutions. Dramatically different time paths of oil use can have essentially the same objective values. It may be that, with the present world economy, an oil cartel could for the present ignore depletion. Similarly, the socially optimal competitive path is unaffected by depletion for many years.

It is generally thought that production paths for monopoly and competitive solutions will cross. This happens here. But cumulative production is always greater in the competitive solution, and the difference grows throughout the competitive period.

The length of time period is frequently assumed in optimization problems, and is usually viewed as years to depletion. In part, this is because of the difficulty of determining numerical answers to the problem in a continuous framework. But suppose that a monopoly has an n-year contract. Will it make radically different decisions than a monopoly that can decide n? Or suppose loss of ownership as a probability question. Does this affect numerical solutions? Of course, both answers are affirmative.

In addition, the technical change issue can be incorporated into the problem, and solved. Alternatively, environmentally linked costs may be represented in the same function, and solved.

1. Fisher's Problem and Optimal Control

As a text for the discussion, Fisher's problem is useful since his Resource and Environmental Economics has attained wide usage. He gives this problem: there are 10 barrels of oil in the ground, the price demand function for oil is $10 - y, cost is a constant $2, and the discount rate is 10%. What is optimal use over two years?

The objectives are to compare social optimality with competitive and monopoly solutions. If all of the oil must be used, the monopolist uses 4.95 barrels in the first year and the remaining 5.05 barrels in the second year. But the competitive solution is socially optimal, and those values are 5.14 barrels in the first period and 4.86 barrels in the second.

If the monopolist need not produce all the oil, the profit maximizing amounts are 4 barrels per year in each period, leaving 2 barrels unused. The socially optimal, competitive solution would, if unconstrained by a resource limit, use marginal cost pricing, and seek to use 8 barrels per period. Since this is not possible, the constrained values for two years (i.e., 5.14 and 4.86) are correct. Table 1 shows these values.

Fisher has specified the number of time periods, the demand function as linear, and cost as constant. It is these assumptions rather than the specific values (i.e., 10 barrels available, 2 periods) which are the key to easy solution. But if we make n, the number of time periods, a control variable, the problem is more complex. In fact, for the monopoly the optimal number of periods is 8 years. In what follows, we find a method of selecting n, and learn that this method is applicable for finding numerical solutions for general continuous problems involving representative world oil data and alternative objectives.

More formally, the problem is to maximize the present value of economic rent by finding the optimal time path of production and the optimal length of time for production.[1/] The time path or trajectory of production

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1. This definition of the problem and first order canonical conditions is based upon Intriligator's discussion, Chs. 11 and 14.

TABLE 1

FISHER'S OIL PROBLEM: 10 BARRELS, 2 PERIODS

Objective	Monopoly Profit	Competitive Social Optimum
Period 0 use, barrels	4	5.14
Period 1 use, barrels	4	4.86
Price, period 0	\$6	\$4.86
Price, period 1	\$6	\$5.14
Total oil use	8	10
Profit present value	\$30.55	\$28.57
Consumers' surplus, present value	\$15.27	\$23.95

Note: Remember cost is a constant \$2 per barrel. Consumers' surplus before discounting is $\int p dy - 2y$. Two periods are assumed.

is controlled by the decision maker to optimize the desired value. Price is then determined by production, as is economic rent. Production is always the rate of change in cumulative production. Admissible values of production must be such that price, production, and rent are always nonnegative.

If all the resource must be used, then cumulative production reaches the original stock at the end of the optimal time. But, of course, if resource exhaustion is not required, then cumulative production must not exceed the original stock, and optimization may mean that some part of the resource is left unused when production ceases.

This is the basic problem for any finite resource, on any scale whether micro (e.g., a small deposit) or macro (e.g., global oil).

First, I state the continuous problem, and then return to Fisher's discrete problem. In a later section, the monopoly profit objective is replaced by social welfare and competitive objectives, and numerical results follow. The continuous problem, initially for a monopoly, is to maximize the present value of profits, V, with respect to n, the length of the production period, and y(t), the production path.

$$\underset{\{y(t),n\}}{\text{maximize}} \qquad V = \int_0^n \frac{p(y)y - \alpha y}{e^{rt}}\, dt;$$

where:

$$p(y) = \beta_2 - \beta_1 y,$$

$$(1) \qquad y(t) = \frac{dX(t)}{dt},$$

$$\int_0^n y\, dt = X(n) \leq S,$$

$$p,\ y \geq 0,$$

$$py - \alpha y \geq 0.$$

Table 2 summarizes the definitions for Eq. (1). For ease of solution, p(y) is linear and cost α is constant for now.

Cumulative production at any time is X(t), a variable reflecting the "state" of remaining resources. Production, y(t), whatever form it may take, must be the rate of change in cumulative production. Original resource stock is S; it cannot be exceeded. However, the not-more-than constraint creates modest difficulties for solutions. Will the optimal y(t) use up all of the resource by n, or will use cease with oil stocks still available? Finally, prices, quantities, and profit must always be non-negative.

TABLE 2

DEFINITIONS OF VARIABLES

Variable	Definition
α	Average cost per unit
β_0	Optimal monopoly production, unconstrained
β_0^*	Substitution parameter
$\beta_0(t)$	With technical change, unconstrained optimal monopoly production
β_1, β_2	Parameters in price demand function
β_3	Competitive socially optimal production, unconstrained
CV	Competitive industry, present value of profit
DV	Discrete V; see V below
$g(t)$	Probability of a profit
H	Hamiltonian function
$h(t)$	Probability of lost profit
Θ	Rate of change in average cost
λ	Dynamic Lagrange multiplier, costate variable
μ	Accumulation factor for regular investment at interest
n	Length of time period
p	Price or price demand function
r	Interest or discount rate
ρ	Sum of interest and risk discount rates
S	Original resource stock
SV	Social value; present value of consumers' surplus
t	Time
V	Present value of monopoly profit
X	Cumulative production
y	Production or production time path
z	Risk parameter

Intuitively, we might prefer to maximize the future accumulated value at the end of the time period, say A(n). But this is simply linked to V: $A(n) = Ve^{rn}$, so maximizing either A or V is equivalent. The maximum monopoly profit is to be found by determining the functional form of y(t), its actual numerical values, and the length of time n.

The best optimal control technique to use with this problem is the maximum principle. This principle asserts that study of a simpler function based upon the relationships in Eq. (1) can provide information useful in finding a solution. The insight is that, by examining first order conditions, the nature of the solution may be determined.[2] The Hamiltonian function takes the function being maximized over time, here the discounted profit function. It includes a term reflecting the rate of change in remaining stock. It is:

$$(2) \quad H = \frac{p(y)y - \alpha y}{e^{rt}} - \lambda y.$$

The λ term is similar to a Lagrangian multiplier, and is a co-state variable: it is co-multiplied with y, the rate of change in the state variable X. It can be viewed as giving the value of the change in discounted profit associated with a small increase in resource endowment. Approximately, λ is dV/dS.

The Hamiltonian provides the first-order conditions to solve Eq. (1):

$$(3a) \quad \text{Either } \frac{\partial H}{\partial y} = 0,$$

$$(3b) \quad \text{or } \frac{\partial y}{\partial t} = 0, \text{ and}$$

$$(3c) \quad \frac{\partial \lambda}{\partial t} = - \frac{\partial H}{\partial X}\ .$$

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2. All of the functions are continuous and differentiable, and the function being maximized (discounted profit in this section) is appropriately concave.

It is still the case that $y = dX/dt$.

Finally, the second-order condition is:

(4) $\frac{\partial^2 H}{\partial y^2} \leq 0.$

Before applying these relationships, I return to the discrete problem in order to show how 8 years is found to be the optimal monopoly period and how y_t is made explicit. The discrete general analogue is summarized with slight modification of Eqs. (1)-(4):

(5a) $\underset{\{y_t, n\}}{\text{maximize}} \quad DV = \sum_{t=0}^{n-1} \frac{p(y)y - \alpha y}{(1 + r)^t}$;

(5b) $\sum_{t=0}^{n-1} y_t = X(n - 1) \leq S,$

(5c) $H = \frac{p(y)y - \alpha y}{(1 + r)^t} - \lambda y.$

DV is the discrete version of net present value monopoly profit. The other constraints, conditions, and definitions in Table 2 still hold. The logical interpretation of these relationships can be difficult since the true optimum may fall between the integer time units.

The Fisher problem values are, as noted,

(6)
$$\begin{aligned} p(y) &= \$10 - y;\ \beta_2 = 10,\ \beta_1 = 1, \\ S &= 10 \text{ barrels original endowment}, \\ r &= 10\% \text{ per year discount rate}, \\ \alpha &= \$2 \text{ per barrel constant cost}. \end{aligned}$$

First, consider the solution when all the resource is used so $X(n) = S$. Applying the first-order conditions, beginning with Eq. (3a):

(7a) $$\frac{\partial H}{\partial y} = \frac{p + \frac{\partial p}{\partial y} y - \alpha}{(1 + r)^t} - \lambda = 0,$$

(7b) $$\frac{\beta_2 - 2\beta_1 y - \alpha}{(1 + r)^t} = \lambda ,$$

(7c) $$y = \frac{-\lambda(1 + r)^t}{2\beta_1} + \frac{(\beta_2 - \alpha)}{2\beta_1} .$$

Apparently, the co-state multiplier does not change over time, since applying Eq. (3c),

(8) $$- \frac{\partial H}{\partial X} = 0 = \frac{\partial \lambda}{\partial t} .$$

Since λ is constant for given parameters like those in Eq. (6), y in Eq. (7c) can be used in the Eq. (5b) definition of cumulative production using all the resource.

(9) $$\sum_{t=0}^{n-1} \left(\frac{-\lambda(1 + r)^t}{2\beta_1} + \beta_0 \right) = S, \text{ where } \beta_0 = \frac{\beta_2 - \alpha}{2\beta_1} .$$

(10) $$n\beta_0 + \frac{-\lambda}{2\beta_1} \sum_{t=0}^{n-1} (1 + r)^t = S.$$

The summation of the interest terms is the familiar $((1 + r)^n - 1)/r$, the accumulation factor for n years of regular investment at interest. Define this as $\mu(n)$; n has to be determined. Consequently, solving for λ in (10),

(11) $$\lambda = \frac{-2\beta_1(S - n\beta_0)}{\mu(n)} , \text{ and}$$

(12) $$y_t = \beta_0 + \frac{(S - n\beta_0)(1 + r)^t}{\mu(n)} .$$

Note that β_0, n, and S determine whether y_t increases or decreases over time. The β_0 term defined in Eq. (9) is simply optimal y in the absence of a resource constraint. It is the usual result of maximizing $py - \alpha y$. Eq. (12), then, says that optimal production in the early years will be closest to the unconstrained level. As time passes, y_t diverges from β_0. The sign and rate of change of this divergence (i.e., $\partial(y_t - \beta_0)/\partial t$) depends upon the numerical values of resource endowment, the optimal production period, and the interest rate. Generally, if the resource constraint applies and n is properly chosen, we may expect y_t to decline over time.

Using Fisher's values given in Eq. (6), the demand function is

$$(13) \qquad y_t = 4 + \frac{(10 - 4n)(1.1)^t}{\mu(n)} \qquad (t = 0, 1, 2, \ldots, n).$$

Eq. (13) shows optimal y when n is known. Now that we know y_t given n, this can be put back into Eq. (5a) and we can find the optimal n.

$$(14) \qquad \underset{\{n\}}{\text{maximize}}\ DV = \sum_{t=0}^{n-1} \frac{(10 - y_t)y_t - 2y_t}{(1.1)^t},$$

y_t as in Eq. (13).

This is easily programmed, and maximum discounted present value is \$51.57, the optimal time period is 8 years (i.e., n = 7), and oil use declines from 2.08 barrels in the initial period to 0.25 barrels in the last period. Price rises from \$7.92 per barrel in the initial year to \$9.75 in the last year. Lambda, from Eq. (11), equals +\$3.85. The interpretation of λ in this discrete format means that, given n = 7, dDV/dS = \$3.85. (Remember that the initial period has t = 0, and n = 7 means an eight year period.)

2. Global, Discrete Oil Use: Monopoly Solution

There are 1.189 trillion barrels of oil remaining, more or less, from the earth's original endowment of 2 trillion barrels. This includes proved reserves as well as oil remaining to be discovered.[3] If the demand elasticity is -0.5 at current annual world oil use of 20 billion barrels,[4] this would mean, for linear demand, p = \$75 - 2.5y. Figuring average world production cost at \$20 per barrel, parameters for a global oil problem are

$$\begin{aligned} p(y) &= \$75 - 2.5y,\ \beta_2 = 75,\ \beta_1 = 2.5, \\ S &= 1189 \text{ barrels remaining oil resource}, \\ r &= 10\% \text{ per year discount rate}, \\ \alpha &= \$20 \text{ per barrel constant cost}, \\ \beta_0 &= 11. \end{aligned} \tag{15}$$

Using equation (12), oil production in year t is

$$y_t = 11 + \frac{(1189 - 11n)(1.1)^t}{\mu(n)} . \tag{16}$$

3. From USGS, Masters, *et al*., 1983 resource estimate, modified for 1983-84 actual use. Other accessible discussions of total oil resources are the *Oil and Gas Journal*, Exxon, Chapman, and Daly, *et al*.

4. Long run price elasticity values include Daly, *et al*.'s -0.73; Pindyck's linear demand values which include -0.33 and -0.90; and -0.30 from Adams and Marquez. Kouris summarizes several studies; for retail gasoline in the U.S., values range from -0.36 to -1.02.

Searching for optimal n is somewhat more time consuming with these values. It is helpful to find an initial value which may be close to the optimum. Suppose expropriation is not a problem and y declines to near zero as is often assumed. Then, by setting $y(t = n) = 0$, an initial n can be found. Taking Eq. (12), and then the global values,

$$y(t = n) = \beta_0 + \frac{r(1 + r)^n(S - n\beta_0)}{(1 + r)^n - 1} = 0, \tag{17}$$

$$n = \frac{S}{\beta_0} + \frac{1}{r} - \frac{(1 + r)^{-n}}{r}, \text{ which gives} \tag{18}$$

$$n = 118.09 - \frac{(1.1)^{-n}}{.1} . \tag{19}$$

Following the simple iteration first used by Hotelling,[5/] n = 118 is the closest integer value. Below, this method is shown to define the unique optimal n for the continuous problem.

The present value of discretely discounted profit is $3.3274 trillion. Oil production and consumption would decline from 11 billion barrels in year "0" to 1.09 billion barrels in year 117 (Figure 1). Price would arise from $47.50 per barrel in the initial period to $74.77 in the final period. (In Figure 1, the superscript d shows optimal path values, and superscripts a, b, c illustrate three non-optimal time paths.)

Given oil use of about 20 billion barrels annually and a price of $27 per barrel in 1985, it seems that monopoly profit maximization was not attained,[6/] at least in the context of these assumptions.

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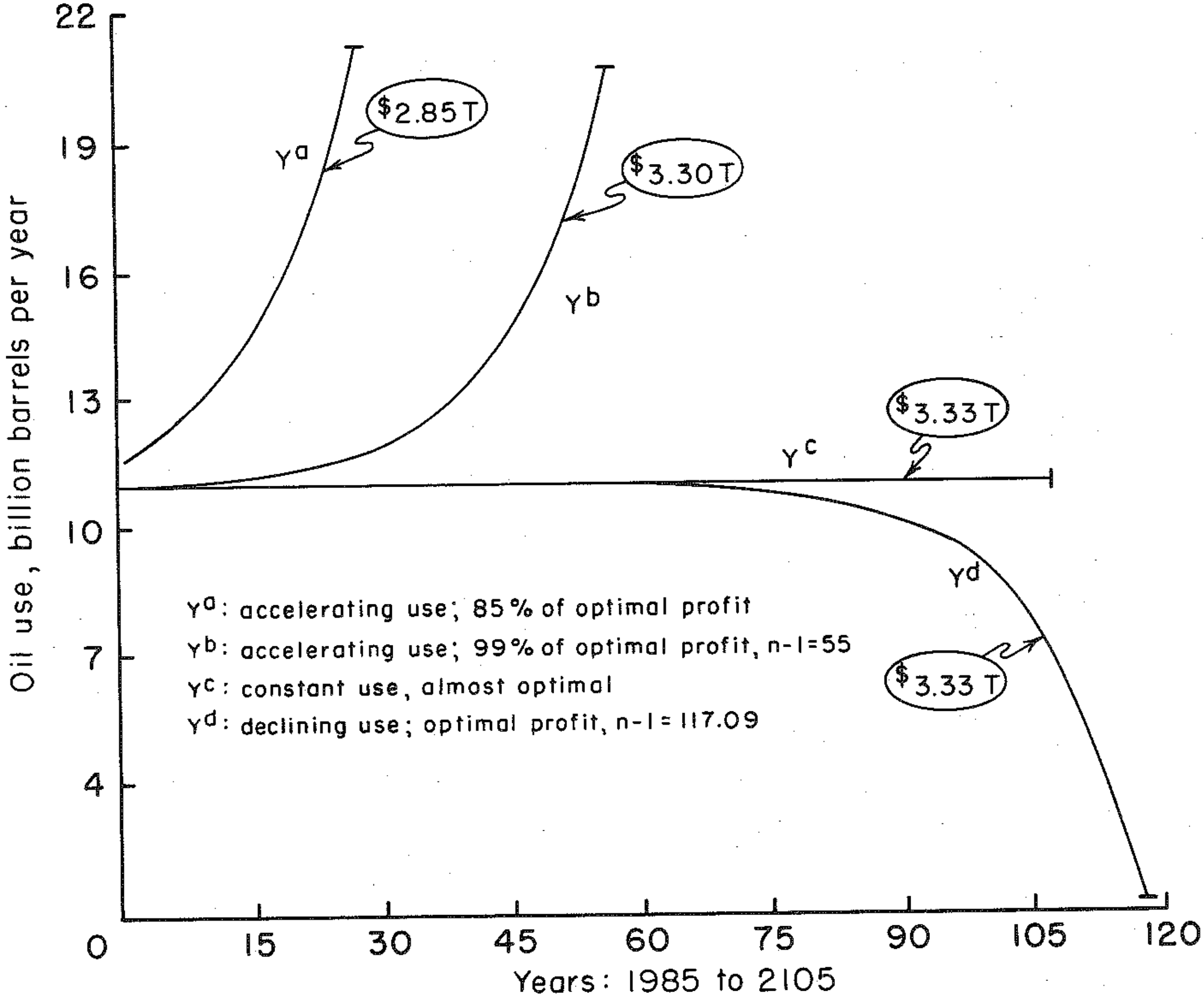
5. Hotelling, p. 142.
6. The $27 per barrel figure represents an average price for all of 1985. See *Weekly Petroleum Status Report*, recent issues.

FIGURE 1. PETROLEUM USE PROFILES AND MONOPOLY PROFIT, trillion dollars



3. Continuous Solutions: Competition, Monopoly, and Social Optima

Although continuous relationships are more easily interpreted, they have not attained frequent use in numerical problems. In part, this is because evaluation of present value V in Eq. (1) can be very difficult. However, new programs such as MACSYMA[7/] now permit resolution of this difficulty. Explicit answers to the Eq. (1) problem for numerical assumptions in Eq. (15) follow.

Beginning with Eqs. (1) - (4), the first order (canonical) conditions are

$$(20) \qquad \frac{\partial H}{\partial y} = \frac{p + \frac{\partial p}{\partial y}y - \alpha}{e^{rt}} - \lambda = 0 \ ,$$

$$(21) \qquad y = \frac{-\lambda e^{rt}}{2\beta_1} + \frac{(\beta_2 - \alpha)}{2\beta_1} \ .$$

As above, since Eq. (3c) leads to $-\partial H/\partial X = 0$, then $\partial\lambda/\partial t = 0$, and λ is constant for a given n. Consequently, production over the period, if it exhausts the resource, is

$$(22a) \qquad \int_0^n \left(\frac{-\lambda e^{rt}}{2\beta_1} + \beta_0 \right) dt = S, \text{ where } \beta_0 = \frac{\beta_2 - \alpha}{2\beta_1} \ , \text{ or,}$$

$$(22b) \qquad \frac{-\lambda}{2\beta_1} \int_0^n e^{rt} dt + \beta_0 \int_0^n dt = S,$$

$$(22c) \qquad \frac{-\lambda}{2\beta_1} \frac{(e^{rn} - 1)}{r} + n\beta_0 = S.$$

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7. MACSYMA is a computer program for algebra and applied mathematics which, because of its calculus capabilities, is particularly useful for solving optimal control values. Rand offers a good program guide, and complete documentation is published by the MIT Mathlab Group and Symbolics, Inc. Microcomputers can use similar packages.

So, with the accumulation factor, now $\mu(n) = (e^{rn} - 1)/r$,

(23) $$-\lambda = \frac{2\beta_1(S - n\beta_0)}{\mu(n)},$$

(24) $$y(t) = \frac{(S - n\beta_0)e^{rt}}{\mu(n)} + \beta_0, \ t = 0, n.$$

This $y(t)$ and λ are identical to the discrete forms for y_t and λ, (Eqs. (11) and (12) above). Clark (p. 146) outlined part of this approach 10 years ago, noting "Determination of T [the optimal exhaustion date], however, is a non-trivial matter." He suggested iterating λ.

Margaret Morgan and William Podulka contributed an explicit solution for the optimal present value of monopoly profit by using $y(t)$ in Eq. (1), and

(25) $$V = \frac{\mu(n)\beta_1\beta_0^2}{e^{rn}} - \frac{\beta_1(S - n\beta_0)^2}{\mu(n)}.$$

Podulka also noted that maximizing V for n gives a unique solution to the problem of optimal n, and that is,

(26) $$n = \frac{S}{\beta_0} + \frac{1}{r} - \frac{e^{-rn}}{r}.$$

This of course provides the basis for the Hotelling iteration for Eqs. (17)-(19). We now know that, for this problem, Eq. (26) is the single unique optimum length of time for production. The value for n is found by simple iteration or by Newton's method,[8] and is 118.09.

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8. Conrad and Clark (Ch. 1.6) show applications of Newton's solution method to optimal control resource problems.

The present value of monopoly profit is $3.025 trillion. This is reassuringly close to the discrete solution. The second order condition [Eqs. (20) and (4)] is

$$(27) \qquad \frac{\partial^2 H}{\partial y^2} = \frac{-2\beta_1}{e^{rt}} < 0 .$$

The same methods find numerical solutions to the socially optimal, competitive problems. Both have the same first-order conditions. The competitive problem is

$$(28) \qquad \underset{\{y(t),n\}}{\text{maximize}} \quad CV = \int_0^n \frac{p(y)y - \alpha y}{e^{rt}} dt, \; \frac{\partial p}{\partial y} \equiv 0,$$

and the social optimum problem is, for social value SV,

$$(29) \qquad \underset{\{y(t),n\}}{\text{maximize}} \quad SV = \int_0^n \left(\int_0^{y(t)} (p(y) - \alpha) dy \right) e^{-rt} \, dt.$$

The competitive and socially optimal Hamiltonians, H_c and H_s, are

$$(30) \qquad H_c = \frac{p(y)y - \alpha y}{e^{rt}} - \lambda y, \; \frac{\partial p}{\partial y} \equiv 0;$$

$$(31) \qquad H_s = \int_0^y \frac{(p(y) - \alpha) dy}{e^{rt}} - \lambda y.$$

For each, the first-order condition is identical.

$$(32) \qquad \frac{\partial H}{\partial y} = \frac{p(y) - \alpha}{e^{rt}} - \lambda = 0.$$

Again, since the first order condition $\partial\lambda/\partial t = -\partial H/\partial X$ is zero, λ is constant for any time period n. Continuing, and using y_c for the socially optimal, competitive path,

(33) $$y_c(t) = \frac{(S - n\beta_3)e^{rt}}{\mu(n)} + \beta_3, \text{ where } \beta_3 = \frac{\beta_2 - \alpha}{\beta_1},$$

(34) $$\lambda_c = \frac{-(S - n\beta_3)\beta_1}{\mu(n)},$$

(35) $$n_c = \frac{S}{\beta_3} + \frac{1}{r} - \frac{e^{-rn}}{r},$$

(36) $$\frac{\partial^2 H}{\partial y^2} = \frac{-\beta_1}{e^{rt}} < 0.$$

As with the monopoly, social value is made explicit with Eq. (33) in Eq. (28), and n_c in Eq. (35) maximizes this social value:

(37) $$SV = \frac{\beta_1\beta_3{}^2\mu(n)}{2e^{rn}} - \frac{(S - n\beta_3)^2}{2\mu(n)}.$$

In comparing monopoly and competition for $\{y_m(t),n_m\}$ and $\{y_c(t),n_c\}$, for these global values, the production and price paths intersect near n_c. The socially optimal path begins at an annual 22 billion barrels, and declines for 64.02 years until the resource is exhausted. The monopoly path begins at 11 billion barrels, stays at this level for many years, and then declines toward zero. Eqs.(38) and (39) summarize these numerical results, as do Figures 2-4.

(38) $$y_m(t) = 11 - \frac{.818\ e^{.1t}}{10{,}000}, \text{ and}$$

(39) $$y_c\ (t) = 22 - .0364\ e^{.1t}.$$

The subscript c indicates competitive, socially optimal solutions, and m indicates monopoly solutions. The price paths do have the expected relationships to the interest rate and opportunity cost:

(40) $$\frac{dp_m}{dt} = .5r\lambda_m e^{rt} > 0,$$

FIGURE 2. PRODUCTION PATHS, SOCIALLY OPTIMAL AND MONOPOLY SOLUTIONS

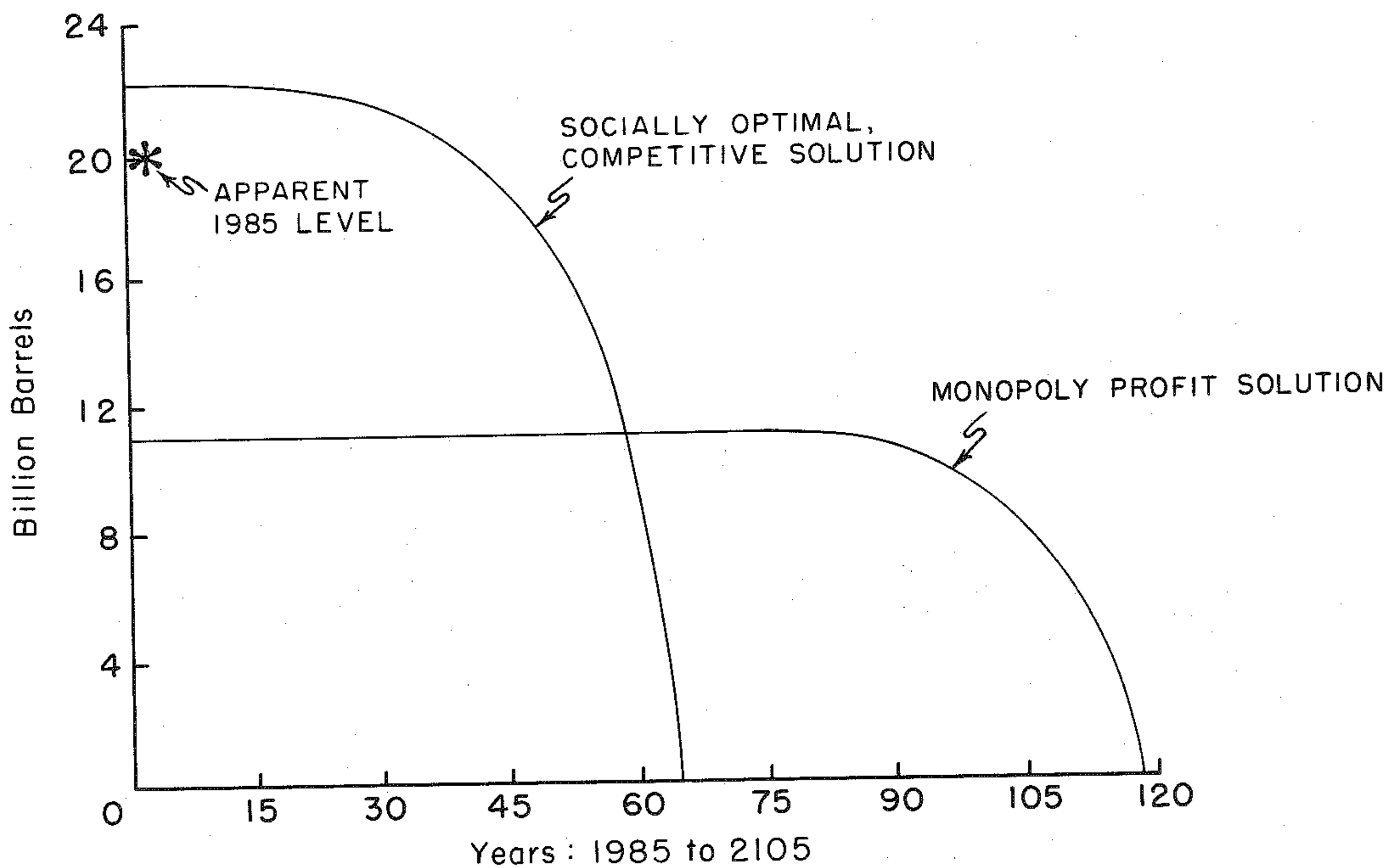


FIGURE 3. CUMULATIVE PRODUCTION, COMPETITION AND MONOPOLY

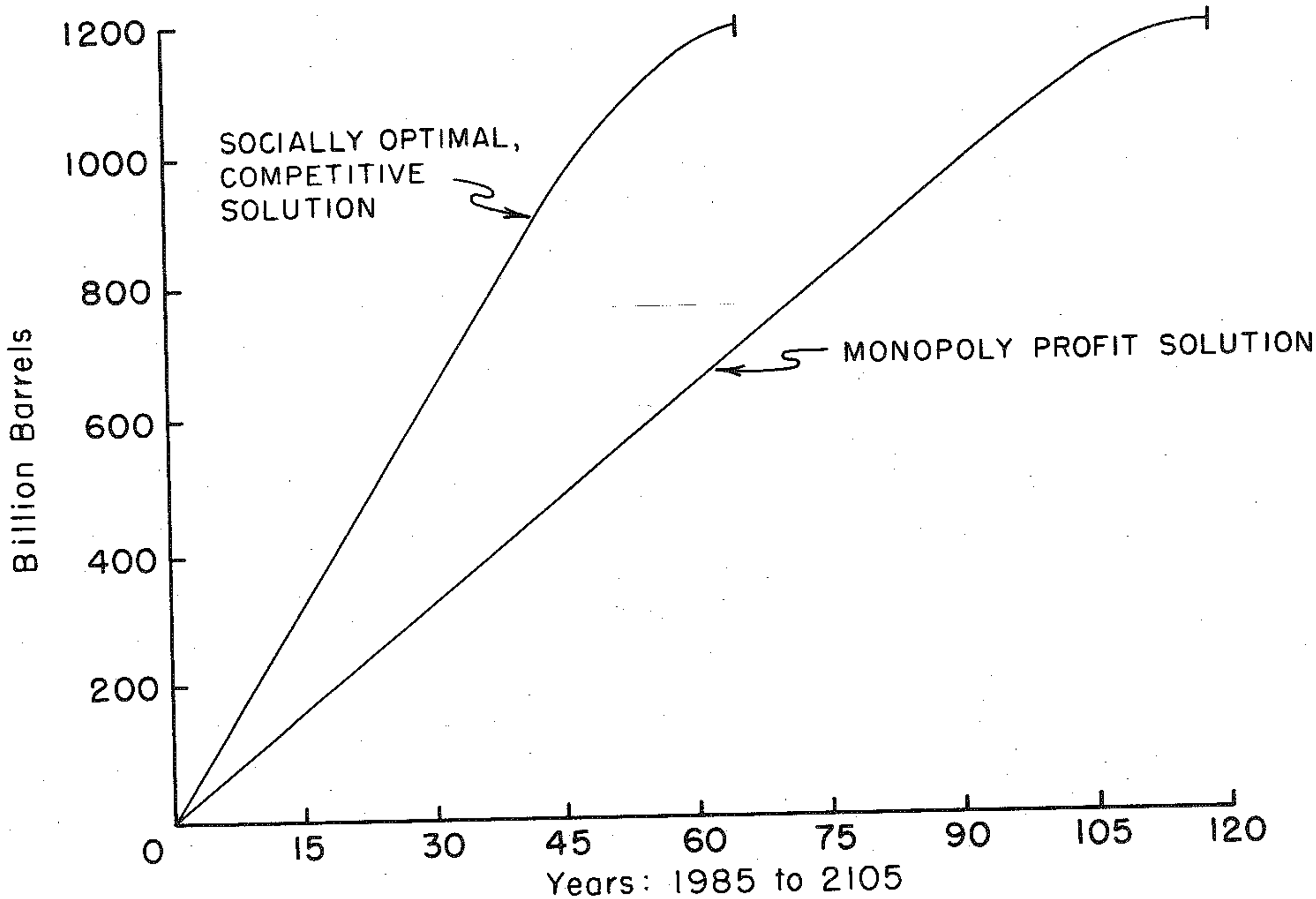
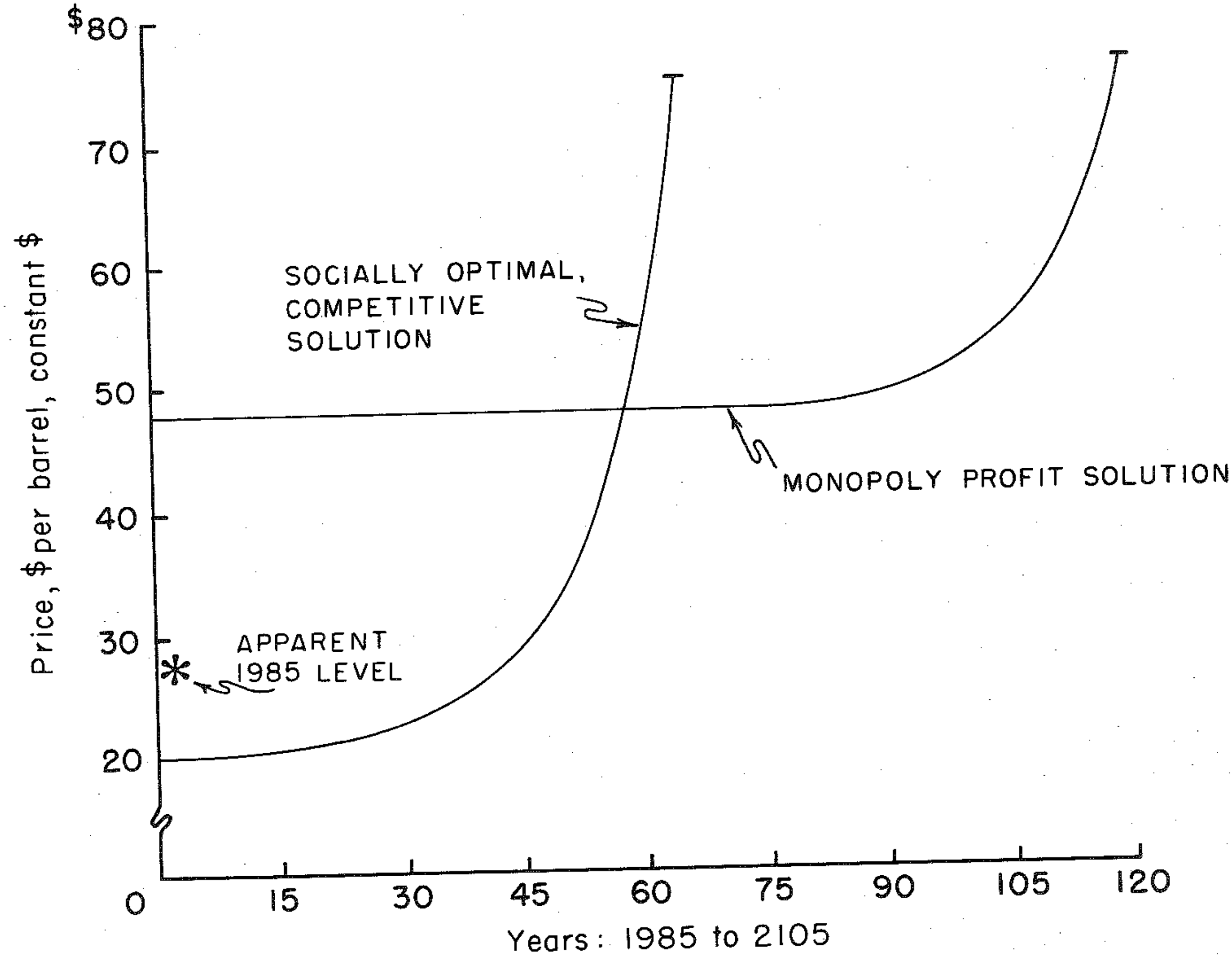


FIGURE 4. PRICE PATHS, SOCIALLY OPTIMAL AND MONOPOLY SOLUTIONS



(41) $$\frac{dp_c}{dt} = r\lambda_c e^{rt} > 0 \; .$$

Here λ_m is .04 of 1¢ and λ_c is 9¢. The depletion periods are considerably different. The implication would be that an increment to world oil resources is more valuable to a competitive world economy than it would be to a world cartel.

4. The Numerical Paradox: Impatience and Myopia are Almost Optimal

Figure 1 had shown that for the discrete problem, production paths that diverge sharply from the optimal can give essentially equivalent profit. It is not surprising that this is equally applicable to continuous solutions for both the social optimum and monopoly problems. Suppose we examine these numerical relationships in a more systematic matter. Will the paradox of widely divergent policies having nearly optimal present value continue? In a general sense, the answer is affirmative, at least for the domain of varied parameters considered here.

Define a myopic planner as setting y at a constant β_3 level until the resource is exhausted after S/β_3 years. (Recall β_3 is the socially optimal level of production in the absence of a finite resource constraint.) Will the present value of net social value be similar to that for optimal y^* for significant variations in interest rate, demand parameters, cost, and oil resources?

Define an impatient planner as wishing to initiate production at the β_3 level, gain a net social value comparable to the true optimum \$6.03 trillion, and leave the oil business. This can be accomplished as follows. First, use a form $y^a(t) = \beta_3 + \beta_4 e^{rt}$. Second, define comparable social value (or monopoly profit) as 95% of the optimum value: $SV^a\{y^a(t),n^a\} = .95\ SV^*\{y^*(t),n^*\}$, or $V^a\{y^a(t),n^a\} = .95\ V^*\{y^*(t),n^*\}$. Now find β_4 which, meeting the non-negativity constraints, gives the least years of production n^a. This is also the maximum β_4 which satisfies the comparability requirement. For Figure 5, for example, with the basic parameters for the maximum social value problem, the impatient planner path is $y^a(t) = 22 + .577e^{.1t}$. Petroleum use increases over a 36 year period. The net social value for this dramatically different path is \$5.74 trillion, 95% of the true optimum.

Figure 6 is showing a case where the basic assumptions are changed significantly in directions intended to reduce this apparent divergence

FIGURE 5. PETROLEUM USE PROFILES AND NET SOCIAL VALUE
Basic values, trillion dollars

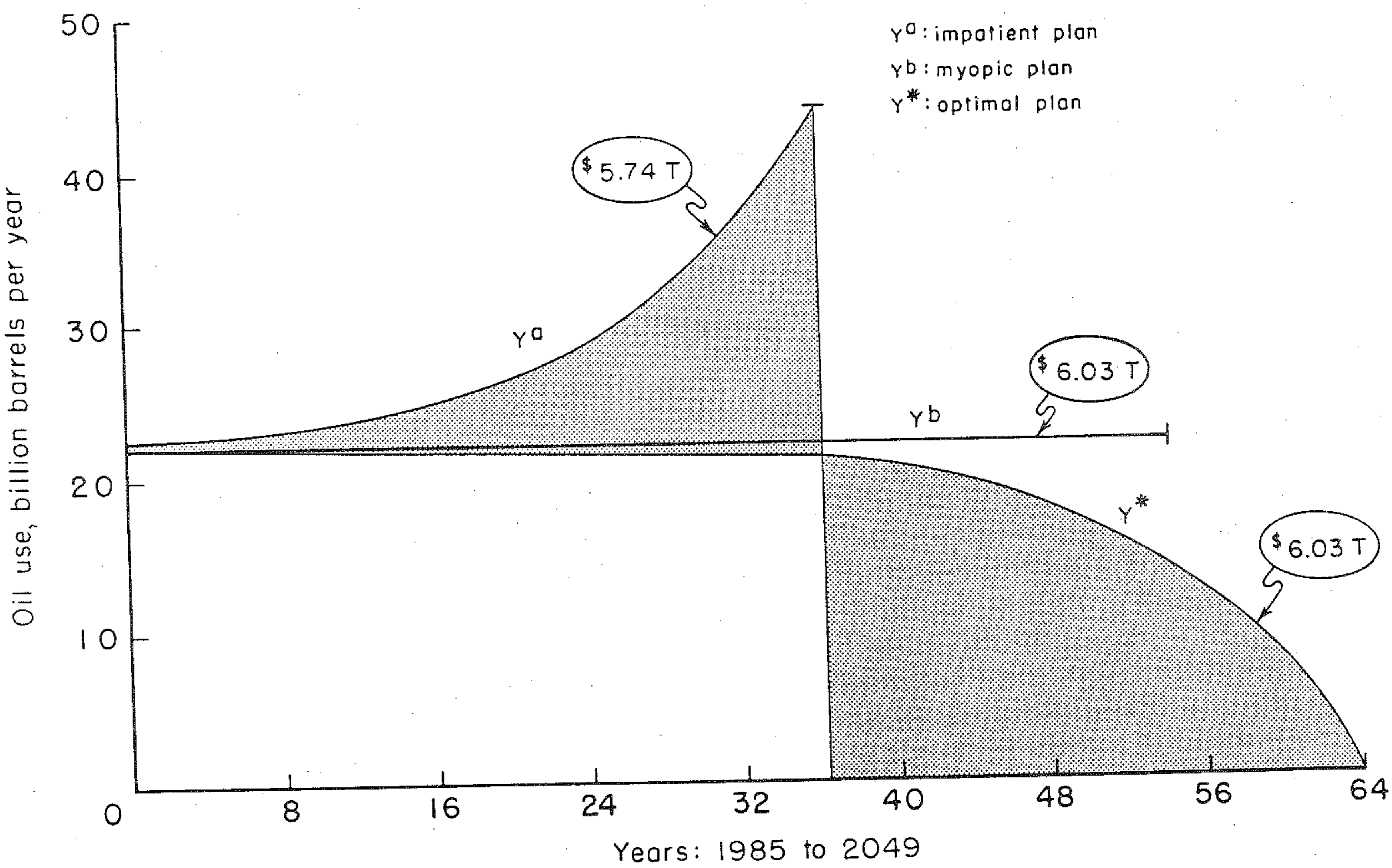
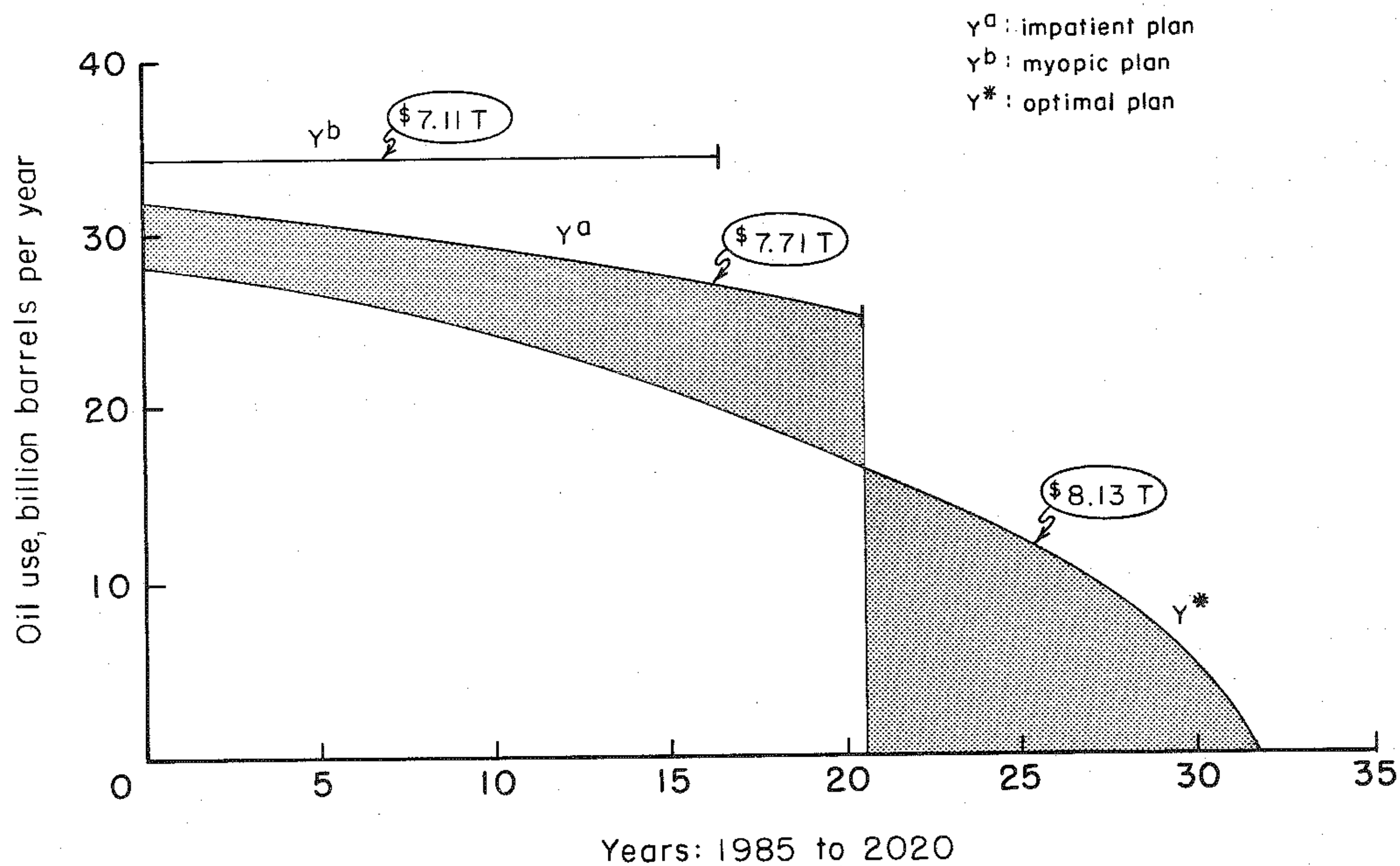


FIGURE 6. PETROLEUM USE PROFILES AND NET SOCIAL VALUE
High elasticity; low cost, interest rate, resources; trillion dollars



in production plans for policies with comparable net social value. Demand elasticity is increased from -0.5 to -1.2, cost is reduced to \$10 per barrel, interest is 5%, and remaining resources are assumed to be one-half of 1189 billion barrels. This combination increases optimal net social value to \$8.13 trillion. The impatient plan Y^a attains 95% of this when the resource is exhausted in 2006.

The implication is that optimal planning may not be necessary to gain either monopoly profit or social value. Strongly divergent production paths are comparably good in terms of objective function numerical values. This is obviously not the case for all possibilities, particularly where the resource is very limited. In the Fisher problem above, for example, the paradox disappears.

A sensitivity analysis indicates that the paradox may apply to a broad domain of parameter values applicable to world oil markets. In Figure 5, the unshaded area represents oil production common to both divergent path Y^a and optimal path Y^*. The shaded area is production not common to both paths. A divergence ratio can be the ratio of the sum of not-common production to the total resource produced on the optimal path. Graphically, this is the ratio of shaded area in Figures 5 and 6 to the appropriate S for each case. Numerically, this is: $\int (|y^a(t) - y^*(t)|)/S \, dt$.[9]

Table 3 shows a sensitivity analysis where divergence ratios are compared for several monopoly and social optimum-competition cases. In each instance, original resources, interest rate, cost, or elasticity are varied by sizable decrements for each market structure. Optimal paths and optimal lengths of production vary accordingly, as does the impatient path Y^a for each case. Figures analogous to Figures 5 and 6 exist for each of the 10 cases. Divergence ratios are noticeably large for all cases.

A similar sensitivity analysis for the myopic planner ($Y^b(t) = \beta_0$ for monopoly, β_3 for social optimum) is not shown, but in eight of the ten cases the value attained or exceeded 95% of the true optimum.

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9. Since Y^a usually accelerates while Y^* declines, and the production period for Y^a is always shorter, a correlation measure of divergence seemed less descriptive.

TABLE 3

DIVERGENCE PARADOX RATIOS

Cases	Monopoly	Competition or Socially Optimal Plan
1. Original global oil parameters, Eq. 15	.75	.53
2. But r = 5%	.53	.41
3. r = 10%, but S = 594.5 billion bl	.53	.41
4. r = 5% and S = 594.5 billion bl	.40	.39
5. Elasticity is -1.2, cost is $10 bl, r = 5%, S = 594.5 billion bl	.39	.40

In early 1986, the average world contract price is $15 per barrel and spot market prices are lower. U.S. petroleum use is increasing as a consequence. A general conclusion is that world oil markets have moved from a mixed competition/cartel combination in 1985 to a competitive market in 1986. The presence or absence of an effective world monopoly is probably much more important in determining annual use, prices, and the years to exhaustion than are specific parameter values.

There is no intention here to determine the empirical parameters of world oil markets and the nature or duration of the current transition to a primarily competitive market. Griffin's recent article on OPEC is considerably more relevant on this point. Solow observed 12 years ago that because optimal n is longer for monopoly than for the competitive social optimum, "the monopolist is the conservationist's friend" (p. 8). However, the numbers show a relationship often obscured. Potential monopoly profit with basic parameters is $3 trillion, one-half of the potential net social value of $6 trillion.

The modest significance here is that numerical analysis with optimal control gives different economic interpretations than would be possible with theory alone, and that dramatically non-optimal paths can have almost optimal objective function values.

5. Qualifications and Conclusions: Risk; Technology Shift and External Social Cost

Fisher's summary (p. 46) of Koopmans, Heal, *et al*. indicates the nature of risk discounting. The virtue of the quadratic form is that risk and probability can be added, and numerical solutions attained. Define the probability that profit in year t is lost through expropriation or other reasons as h(t), and have that probability increase over time. The probability of a profit is g(t), which decreases over time.

(42a) $h(t) = 1 - e^{-zt},\ 0 \leq z \leq 1;$

(42b) $g(t) = 1 - h(t) = e^{-zt}$.

Then, the expected profit EV is modified from Eq. (1) as

(43) $$\underset{\{y(t),n\}}{\text{maximize}}\ EV = \int_{o}^{n} \frac{p(y)y - \alpha y}{e^{rt}}\, g(t)\, dt = \int_{o}^{n} \frac{p(y)y - \alpha y}{e^{\rho t}}\, dt .$$

So, by simply introducing z into "risk discounting," the denominator in EV is replaced by $e^{\rho t}$, and ρ is simply r + z. For example, for our numerical solutions, suppose r = 9.5% annual interest and z = .005 for the risk term. Since ρ is the same 0.1, all the numerical solutions are identical.

Technological change can be represented simply by a factor such as $e^{\theta t}$, with θ being the rate of change in average cost. If $\alpha = \alpha_0 e^{\theta t}$, then $(\partial\alpha/\partial t)/\alpha = \theta$. Speculating, θ may have been negative between 1859 and 1970, and positive now.

Also, a growing recognition of external social cost and its internalization may be taking place, causing θ to be positive.

Eq. (1) is restated, and the Hamiltonian and first order conditions and solution follow for monopoly profit.

(44) $$\underset{\{y(t),n\}}{\text{maximize}}\ VC = \int_{o}^{n} \frac{p(y)y - (\alpha_0 e^{\theta t})y}{e^{rt}}\, dt ,$$

(45) $$H = \frac{p(y)y - (\alpha_0 e^{\Theta t})y}{e^{rt}} - \lambda y \; .$$

The derivation follows the exposition used above, and the optimal production path becomes

(46) $$y(t) = \beta_0(t) + \frac{(S - n\beta_0^*)e^{rt}}{\mu(n)} \; , \; t = 0, n;$$

(47a) $$\beta_0(t) \equiv \frac{\beta_2 - \alpha_0 e^{\Theta t}}{2\beta_1} \; , \text{ and}$$

(47b) $$\beta_0^* \equiv \frac{\beta_2}{2\beta_1} - \frac{\alpha_0(e^{\Theta n} - 1)}{\Theta n 2\beta_1} \; .$$

The exact solution for $y(t)$ in Eq. (46) is the same as Eq. (24) except for the technical change impact in $\beta_0(t)$ and β_0^*. Finding n is somewhat more complex, but analogous to the earlier methods. The result is

(48) $$n = \frac{S}{\beta_0^*} + \frac{(1 - e^{-rn})}{r} \; \frac{\beta_0(n)}{\beta_0^*}$$

If there is no technical or environmentally induced cost change and $\Theta = 0$, then both $\beta_0(n)$ and β_0^* are simply β_0. Eq. (48), then, becomes Eq. (26).

Suppose $\Theta = .0462$ so that average cost doubles to \$40 per barrel by 2000 and quadruples by 2015. The numerical solution for $y(t)$ requires attention to interpreting the constraints in Eq. (1). Non-negative profit requires production to cease in 28.60 years when cost exceeds price, and Eq. (48) is inapplicable. So $n^* = 28.60$ and

(49) $$y(t) = \beta_0(t), \; t = 0, 28.60.$$

Here, $\beta_0(t)$ represents time dependent optimal production with rising cost in the absence of an effective resource constraint. The monopolist ceases production in 2013, when cost reaches \$75. Cumulative use is 292 billion barrels.

The rate of cost increase must be no greater than .743 of 1% for all of the 1189 billion barrels to be used. Suppose $\theta = .005$. Then Eqs. (46)-(48) are applicable. Eq. (48) gives 135.8 years as one solution. The integral for present value Eq. (44) is considerably more complex. However, using the MACSYMA program again, the optimum monopoly profit is \$2.9 trillion.

In other words, if cost shift is rapid, the optimal production period is shorter, and there may be unused resources when production ceases.

To recapitulate, finding numerical values for the optimal length of a production period has been difficult, and finding numerical values for economic rent and social value functions has been nearly impossible. With computer-assisted algebra and analytic solutions, the process can be shortened to minutes. The steps are (a) define the first-order canonical conditions with discounting explicit, (b) express y, the equilibrium quantity, as a function of the parameters for demand, cost, and the co-state multiplier, (c) solve for the co-state multiplier λ as a function of the resource stock, the demand and cost parameters, and the interest accumulation factor, (d) solve for y, now an explicit function of n, t, the resource stock, and the demand and cost parameters, (e) find optimal n, (f) evaluate the objective function, and (g) evaluate the second-order conditions.

The same general technique should be followed for possible optima in which all of the resources need not be used. In both cases be sure that only permissible values of quantities, prices, and profit are examined. Compare as appropriate the constrained optimum path and length of time period in which all of the resource must be used with the unconstrained optimum.

These steps lead quickly to numerical answers for different decision goals, whether that be monopoly profit, consumers' surplus, or competitive equilibria. Both continuous and discrete formulations are easily solved, and the method is easily extended to include the risk of expropriation, and a technical change/environmental cost continuous shift. With these problems capable of being solved in minutes with MACSYMA and other computer algebra, classroom use and applied research may both be expected to grow at exponential rates.

6. References

Adams, F. Gerard; and Jaime Marquez. 1984. "Petroleum Price Elasticity, Income Effects, and OPEC's Pricing Policy." Energy Journal 5(1): 115-128.

Chapman, Duane. 1983. Energy Resources and Energy Corporations. Ithaca, NY: Cornell University Press.

Clark, Colin W. 1976. Mathematical Bioeconomics. New York, NY: Wiley and Sons, Inc.

Conrad, Jon M.; and Colin W. Clark. forthcoming. Notes and Problems in Resource Economics. Cambridge, UK: Cambridge University Press.

Daly, George; James M. Griffin; and Henry B. Steele. 1982. "Recent Oil Price Escalations: Implications for OPEC Stability." In OPEC Behavior and World Oil Prices. Griffin and Teece, David J.; eds. Boston, MA: Allen & Unwin, Inc.

Exxon Corporation. 1982. "Improved Oil Recovery."

Fisher, Anthony C. 1981. Resource and Environmental Economics. Cambridge, UK: Cambridge University Press.

Griffin, James M. "OPEC Behavior: A Test of Alternative Hypotheses." American Economic Review 75(5): 954-963.

Hotelling, Harold. 1931. "The Economics of Exhaustible Resources." Journal of Political Economy 39(2): 137-175.

Intriligator, Michael D. 1971. Mathematical Optimization and Economic Theory. Englewood Cliffs, NJ: Prentice-Hall, Inc.

Kouris, George. 1983. "Energy Demand Elasticities in Industrialized Countries." Energy Journal 4(3): 73-94.

Mathlab Group. 1983, revised. MACSYMA Reference Manual. 2 vols. Cambridge, MA: MIT Laboratory for Computer Science, and Symbolics, Inc.

Morgan, Margaret. 1985. Unpublished seminar paper.

Oil and Gas Journal. Dec. 31, 1984.

Pindyck, Robert S. 1978. "Gains to Producers from the Cartelization of Exhaustible Resources." Review of Economics of Statistics 60: 238-251.

Podulka, William. 1985. Unpublished seminar paper.

Rand, R.H. 1984. Computer Algebra in Applied Mathematics: An Introduction to MACSYMA. Boston, MA: Pitman.

Solow, Robert M. 1974. "The Economics of Resources or the Resources of Economics," American Economic Review 64(2): 1-14.

U.S. Department of Energy. Weekly Petroleum Status Report.

USGS: U.S. Geological Survey. C.D. Masters et al. 1983. "Distribution and Quantitative Assessment of World Crude-Oil Reserves and Resources." Reston, VA: Open-File Report 83-728.

______. Charles D. Masters. 1985. "World Petroleum Resources - A Perspective." Reston, VA: Open-File Report 85-248.