Choice in Risk Situations: A Comparison and Evaluation of Alternative Criteria

by

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PREFACE

Research concerned with decision making in situations where the consequences of a particular action are not known with certainty has increased tremendously over the past twenty years. These efforts have taken the form both of developing new theories of choice and applying new or existing theories to individual and firm decision problems. Particular emphasis has been placed on the empirical application to portfolio analysis and farm production decisions.

The decision criteria which have been brought to bear on the above decision problems have logically been those which lend themselves to mathematical programming analysis. This paper compares and evaluates several decision criteria which can be applied empirically with the help of linear or quadratic programming, although little reference is made to how the application is accomplished. The paper emphasizes and evaluates the theoretical and empirical evidence which has been given in support of these various criteria.

The paper had been developed primarily for background reading in a graduate course in quantitative methods. It represents an effort to pull together and evaluate, in one article, the contributions of many scientists, but should not be considered as an exhaustive survey of all recent developments in the field. Attention is focused on those decision criteria whose application can be accomplished with the help of linear and nonlinear programming methods.

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CHOICE IN RISK SITUATIONS:

A COMPARISON AND EVALUATION OF ALTERNATIVE CRITERIA*

by

Richard N. Boisvert

For many years some economists have maintained that decisions made by managers of economic units depend largely on how managers view risk and uncertainty. Since these views of risk and uncertainty vary among managers, decision rules employed and ways of making decisions are also thought to vary among managers. The unique nature of a manager's decision processes presents a substantial obstacle to anyone undertaking a study of decision making in an imperfect knowledge situation that has general applicability. The success of such a study depends, among other things, on developing a theory of choice. This theory must be an abstraction and is unlikely to be identical with any particular decision process in the real world. Nevertheless, the theory must recommend behavior similar to that which would actually be observed by decision makers faced with the same problem and information.

The author believes that although the decision process is unique to each decision maker, there exist certain factors (the expected value of income serves as a good example) which play an important role in the decision process of most economic decision makers. If a number of factors of common concern to decision makers can be identified, it may be possible to develop a theory of choice which is consistent with observed behavior.

*The author wishes to express his appreciation to Doyle Eiler for his helpful comments on an earlier draft of this paper.

The purpose of this paper is to outline a "theory of risky choice" which, when applied to farm planning models will recommend decisions consistent with farm managers' behavior.[1/] Both theoretical and empirical evidence in support of the "theory" proposed are offered throughout the discussion.

ALTERNATIVE THEORIES OF RISKY CHOICE

Because of the complex and unique nature of decision making processes, it is not difficult to understand why people have developed a number of theories of risky choice. By examining these theories and their shortcomings, one is able to evaluate effectively any particular theory of choice.

EXPECTED PROFIT

An early attempt to explain behavior under risk was the theory that people acted so as to maximize expected gain or profit in money terms. This criterion is a logical extension of the theory of choice under certainty. However, several serious objections exist to its acceptance. The first objection is that

> "the ordinary man has to consider the possible outcome of an action on one occasion (or a small number of occasions) only and the average (or expected) outcome, if the conduct were repeated a large number of times under similar conditions, is irrelevant."[2/]

1/ Following R. D. Luce and H. Raiffa, Games and Decisions, (New York: John Wiley and Sons, Inc.), 1957, a risk situation is defined as a situation whose probability of occurrence is known to the decision maker. For example a financial investor whose income varies according to his choice of portfolio and/or random fluctuations in the rate of return on any particular asset in the portfolio, and a farmer whose income depends on the crops he raises and/or random fluctuations in the prices he receives are both decision makers faced with risk situations.

2/ A. D. Roy, "Safety-First and the Holding of Assets," Econometrica, July 1952, p. 431.

In other words only if the action could be repeated an infinite number of times could the decision maker have confidence that his expected gain would approach the true population expectation.

Because this theory implies that a person's marginal utility of income is constant, many people consider it too unrealistic -- a second objection. As a result the expected return criterion fails to explain the behavior of a person who buys insurance or refuses to stake a fortune on the St. Petersburg game (a game whose expected gain is infinite).[1/] Furthermore, this criterion fails to explain the diversification of an investors' portfolio. In other words, a financial investor who wants to maximize expected profit will invest all his funds in that financial asset whose expected net return is greatest.[2/] This behavior is certainly inconsistent with observed financial portfolios.

Although expected profit maximization may be unacceptable as a theory of choice, one should not conclude that the expected profit is uninfluential in decision making. At a later point a theory of risky choice which depends on expected profit and one or more indicators of the dispersion or profit is developed.

EXPECTED UTILITY MAXIMIZATION

The expected utility maximization approach to decision making under risk is an attempt to build a preference for risk aversion into a theory of choice and to explain the fact that diversification is often observed in economic behavior. This approach like the expected profit maximization criterion, can be thought of as an extension of the theory of choice

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1/ For a complete discussion of both the "paradox" of insurance and the St. Petersburg Paradox see D. E. Farrar, The Investment Decision Under Uncertainty, (Englewood Cliffs, N. J.: Prentice Hall, Inc.), 1962, pp. 7-11.

2/ See J. Tobin, "Liquidity Preference as Behavior Toward Risk," Review of Economic Studies, February 1958, for proof of this proposition.

under certainty.

Rational Behavior

Economists define rational behavior in terms of how they believe a "rational" man acts. This rational man resembles an ordinary individual in that he is neither omnipotent nor omniscient. Since his powers are limited and he has limited information, his actions may be less than perfect. However, the rational man is unlike an ordinary person in that every action is perfectly calculated.[1/] For a man to be rational, his decision-making must conform to a set of postulates or axioms such as the following:

Axiom 1: If P and Q are any two probability distributions of outcomes, then either P is preferred to Q, or Q is preferred to P, or both are considered equally good;

Axiom 2: If P is considered at least as good as Q and Q is considered at least as good as R, then P is considered to be at least as good as R;

Axiom 3: If probability distribution P is preferred to probability distribution Q, and if R is any probability distribution at all, then a probability "a" of obtaining P and (1 - a) of obtaining R is preferred to a probability "a" of obtaining Q and (1 - P) of obtaining R -- as long as "a" is not zero;

Axiom 4: If P is preferred to Q and Q is preferred to R, then there must exist a number C such that CP + (1 - C)R is exactly as good as Q.[2/]

Since this rational man does not exist, a study of how he behaves can only serve to formulate general principles by which alternative decision criteria can be judged. The extent to which these principles of rational behavior can be used to rationalize the choice of a particular

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1/ H. M. Markowitz, Portfolio Selection: Efficient Diversification of Investments, (New York: John Wiley and Sons Inc.), 1959, p. 206.

2/ This particular set of axioms is found in H. M. Markowitz, op. cit. pp. 229-235. Markowitz also discusses two variations of axiom 4.

decision criterion is subject to much controversy. Unfortunately, these principles, along with a limited number of empirical tests of decision criteria and some common sense, provide the entire set of norms by which one can evaluate alternative decision criteria.

According to this concept of rationality there is but one decision criteria which is rational. This criterion is the expected utility maxim. In other words:

> "If a set of preferences is in accord with the expected utility maxim, it is consistent with the axioms. An individual acts according to the axioms if and only if he acts according to the maxim. If we understand the conditions and requirements imposed by the axioms we understand the assumptions behind the use of expected utility."1/

Acceptance of the expected utility maxim as a theory of risky choice is based on considerations other than the fact it is consistent with the economists' concept of rationality. Friedman and Savage, for example, demonstrate that an individual may have aversion to some risks and no aversion to others and still behave according to the expected utility maxim.2/ They are in fact able to reconcile gambling by a person who has a general predominance for risk aversion -- a condition which must hold if the utility function is to be bounded from above and below.3/ The maxim,

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1/ The quotation and the proof of its validity (for utility functions bounded both from below and above, i.e., letting utility depend on a variable Y and $U(Y)$ = total utility of Y, boundedness requires $\lim U(Y)$ as $Y \to 0$ and $Y \to \infty$ exist and are finite) are found in Markowitz, op. cit., pp. 18-27.

2/ M. Friedman and L. J. Savage, "The Utility Analysis of Choices Involving Risk," Journal of Political Economy, 1948, pp. 279-304.

3/ A risk averter, according to K. J. Arrow, Aspects of the Theory of Risk Bearing, Helsinki: Yrjo Jahnssonin Saatio, 1965, is defined as a person who, starting from a position of certainty, is unwilling to take a bet which is actuarially fair. A risk lover is someone who is willing to accept an unfair bet. Arrow also discusses the risk aversion hypothesis and its ability to explain this otherwise puzzling behavior.

combined with the risk aversion hypothesis, also serves as a qualitative explanation of observed aversions toward risk which had previously puzzled those people who developed the expected profit maxim:

> "The most obvious is insurance, which hardly needs elaboration. Common stocks with limited liability to the stockholders find a market because of risk aversion. The cost-plus and other forms of risk sharing contracts are again explicable only on the same hypothesis . . . Finally, . . ., the holding of money depends in part . . . on the motive of avoiding risk."[1/]

Furthermore, Tobin has demonstrated that under certain conditions the expected utility maxim can explain the investment in a diversified portfolio.[2/]

Choice of a Utility Function

From a theoretical point of view, the acceptance of the expected utility maxim is very appealing. It has been pointed out that a wide variety of seemingly inconsistent behavior (ranging from gambling and buying insurance to diversification) can all be explained by adherence to the expected utility maxim. Whether a person is a gambler or risk averter depends upon the shape of his utility function. Because the shape of the utility function determines a decision maker's attitude toward risk, a researcher must be concerned with the selection of an appropriate utility function when applying the expected utility maxim to an empirical analysis.

Let a person's utility be a function of his net money income. Then one can write:

R = net income

$U(R)$ = total utility of income.

1/ Ibid.

2/ J. Tobin, op, cit., contains a simple proof of this proposition.

If one assumes that the utility function is at least twice differentiable then one can define:

$U'(R)$ = marginal utility of income (assumed to be non-negative); and

$U''(R)$ = rate of change of the marginal utility with respect to income.

For a person to be a risk averter it is both necessary and sufficient for

(1) $U'(R)$ is a decreasing function of R.1/

Similarly, a risk lover is characterized by

(2) $U'(R)$ is an increasing function of R.2/

It is also possible for a person to exhibit the behavior of a risk averter over some range of income and that of a risk lover over another range -- in which case

(3) $U'(R)$ is an increasing function of R for $R_0 < R < R_1$ and $U'(R)$ is a decreasing function for $R \leq R_0$ and $R \geq R_1$.

These three relationships are illustrated in figure 1.

Deciding upon the appropriate form of the utility function for an

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1/ For risk averters, K. J. Arrow, op. cit., pp. 30-35, suggests that the utility functions should display decreasing absolute risk aversion, but increasing relative risk aversion. Absolute risk aversion is defined as:

$$R_A(R) = -U''(R)/U'(R).$$

Saying that a utility function exhibits decreasing absolute risk aversion amounts to "saying that the willingness to engage in small bets of fixed size increases with wealth, in the sense that the odds demanded diminish," (i.e., the absolute risk aversion decreases as R increases). Relative risk aversion is defined as:

$$R_R(R) = -RU''(R)/U'(R).$$

That is, "if both wealth and the size of the bet are increased in the same proportion, the willingness to accept the bet (as measured by the odds demanded) should decrease," (i.e., the relative risk aversion increases as R increases).

2/ J. Tobin, op. cit., p. 75.

empirical study is difficult because each decision maker operates according to his own unique utility function. Officer and Halter have employed several models for estimating utility functions of individual farmers.[1/] First they determined empirically a number of points on a decision maker's utility surface. Second, they fitted an equation to the observations using monetary outcomes as the independent variable.

Three techniques to generate the individual observations are commonly used. The best known procedure was developed by von Neumann and Morgenstern.[2/] Their technique (N-M) rests on the continuity assumption. That is, if outcome X_1 is preferred to X_2 and X_2 is preferred to X_3, then there exists a probability $p > 0$ such that

$$pu(X_1) + (1 - p)u(X_3) = u(X_2),$$

where $u(X_i)$ is the utility of X_i. X_1, X_3 and their corresponding utilities are set arbitrarily. X_2 is determined by the respondent, after which one can solve for $u(X_2)$. Similarly, as many points on the utility function as desired may be obtained. This procedure, however, has two weaknesses. First, if the subject does not fully understand the concept of probability or has probability preferences, his subjective probabilities may not be the same as the objective probabilities specified in the experiment. Second, because he is given the choice between a gamble and a certain event, the subject's attitude toward gambling may bias the outcome. The remaining two techniques, the modified von Neumann-Morgenstern (MN-M) and the Ramsey model, attempt to overcome these criticisms. The (MN-M) model overcomes the first objection by using neutral probabilities,

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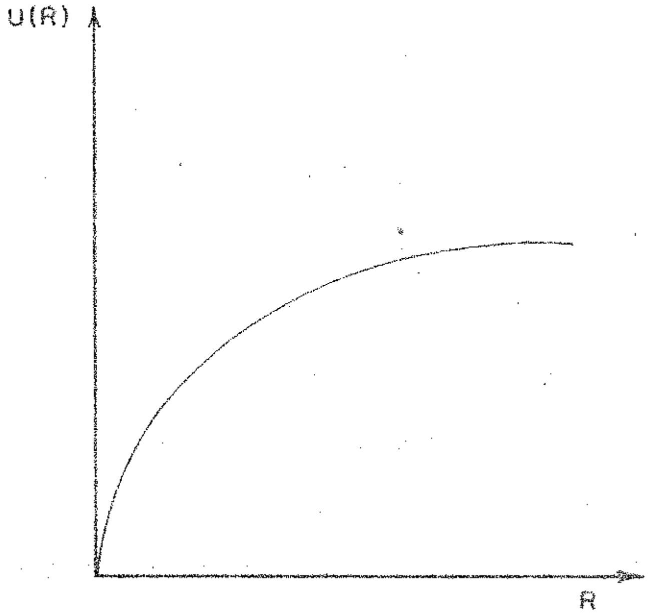
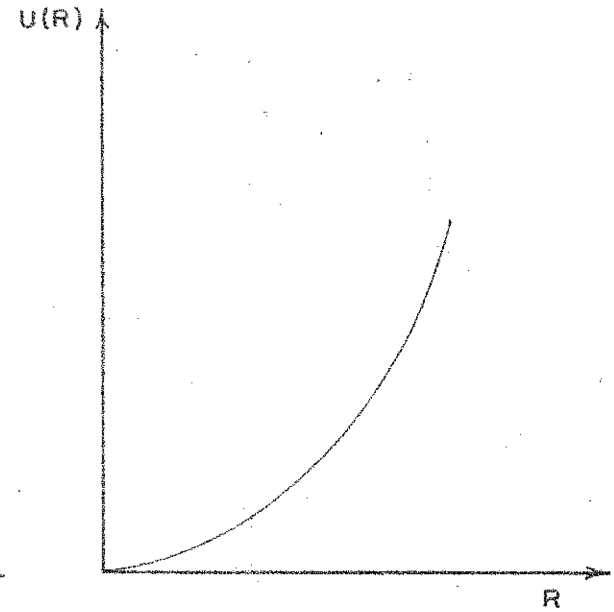
1/ R. R. Officer and A. N. Halter, "Utility Analysis in a Practical Setting," American Journal of Agricultural Economics, May 1968, pp. 255-277.

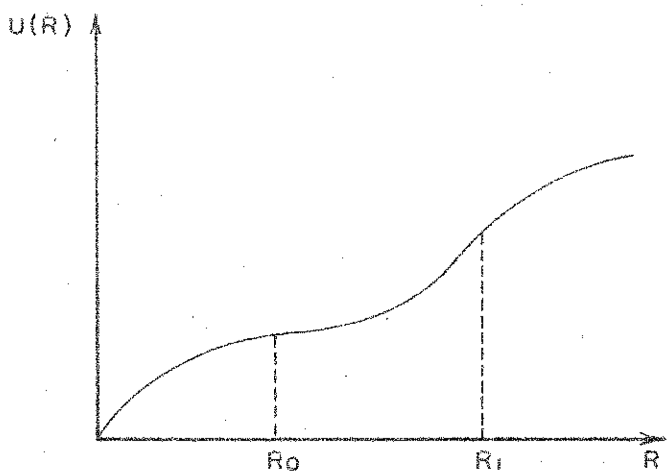
2/ J. von Neumann and O. Morgenstern, Theory of Games and Economic Behavior, (Princeton, N. J.: Princeton University Press), 1947.



1a. Risk Averter



1b. Risk Taker



1c. Risk Taker for $R_0 < R < R_1$
Risk Averter elsewhere

Figure 1. Attitudes Toward Risk

$p = (1 - p) = 0.5$. And, the Ramsey model overcomes both objections by using neutral probabilities and requiring the subject to select between two gambles.[1]

Once the observations on the utility surface have been generated, one can proceed to determine the shape of the respondent's utility function by fitting alternative functions (i.e., linear, quadratic, cubic, etc.) to the data. However, relying on empirically estimated utility functions has two serious limitations. First, the estimates are "individual" specific and research based on them may not generalize to other farm situations. Second, the estimated functional forms may be difficult to incorporate into complex decision models.[2] In other words, to apply the expected utility maxim in many complex decision situations one must assume the decision maker acts according to a utility function the expected value of which is mathematically tractable.[3] Therefore, the utility functions which can be used in decision models involving mathematical programming are few in number. In addition, they have the common characteristic that their expectations depend on the mean income and variance of income.

One possible utility function from which one can choose is, unfortunately, useful only when income is normally distributed. The utility

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[1] See Officer and Halter, op. cit., and R. P. Ramsey, The Foundations of Mathematics and Other Logical Essays, ed. R. B. Braithwaite, (London: Routledge and Kegan Paul Ltd.), 1931; for details.

[2] In decision situations where there is a relatively small number of alternatives from which the decision maker must choose, one is less limited in his selection of utility functions. In these situations the expected utility associated with each alternative can be determined and the alternative which yields the largest expected utility can be chosen. An example where farmers were faced with the relatively simple decision of selecting among 19 fodder reserve programs is given in A. N. Halter and G. W. Dean, Decisions Under Uncertainty with Research Applications, (Chicago: South-Western Publishing Co.), 1971, pp. 72-82.

[3] A function is said to be mathematically tractable if analytical methods exist through which the extreme values of the function can be determined.

function is:[1]

$$(4) \quad U(R) = 1 - e^{-aR}$$

The constant "a" indicates the decision maker's attitude toward risk. Large values of "a" correspond to more conservative decision makers. Freund demonstrates that if net revenue is normally distributed then

$$(4a) \quad E[U(R)] = \int_{-\infty}^{\infty} (1 - e^{-aR}) e^{-(R-\mu)^2/2\sigma^2} dR$$

and its maximization is equivalent to maximizing

$$(4b) \quad \mu - \frac{a}{2}\sigma^2.$$

One undesirable feature of this particular utility function is that it is concave everywhere and therefore, exhibits risk aversion everywhere. It also exhibits increasing absolute risk aversion -- a property that Arrow finds objectionable.

If one makes no assumption about the probability distribution of R but assumes a decision maker's utility function is

$$(5) \quad U(R) = (1+b)R + bR^2, \; b < 0 \text{ and } (1+b) + 2bR \geq 0 \text{ (so that}$$

$U'(R) \geq 0$) then the maximization of

$$(5a) \quad E[U(R)] = \int_{-\infty}^{\infty} [(1+b)R + bR^2] f(R) dR$$

is accomplished by maximizing a quadratic equation[2]

$$(5b) \quad E[U(R)] = (1+b)\mu + b(\sigma^2 + \mu^2).$$

1/ R. J. Freund, "Introduction of Risk into a Programming Model," Econometrica, July 1956, pp. 253-263.

2/ J. Tobin, op. cit., p. 77.

This particular utility function is also concave everywhere and does not escape Arrow's objection either.

Farrar takes a much different approach (one which also does not depend on the particular probability distribution) but develops an "expected utility" decision model also involving the maximization of a quadratic equation. He begins by assuming a person's utility of money function is: 1) positively sloped and concave downward and 2) at least twice differentiable in the vicinity of its mean.[1/] Under these assumptions, he attempts a quadratic approximation to the utility function.

The utility function can be expanded by a Taylor's series about the mean

$$(6) \quad U(R) = U(\mu) + U'(\mu)(R-\mu) + \frac{U''(\mu)}{2}(R-\mu)^2 + \frac{U'''(\mu)}{3!}(R-\mu)^3 + \ldots$$ [2/]

Dropping all terms beyond the quadratic and applying the expected value operator, the investor's expected utility may be expressed as

$$(7) \quad E[U(R)] = U(\mu) + U'(\mu)E(R-\mu) + \frac{U''(\mu)}{2}E(R-\mu)^2.$$

Recalling that $E(R-\mu)^2$ is simply the variance of R and $E(R-\mu) = 0$, the middle term vanishes and

$$(8) \quad E[U(R)] = U(\mu) + \frac{U''(\mu)}{2}\sigma^2.$$

Under a number of reasonable assumptions an investor's utility function (and its expected value) is unique up to a linear transformation. We

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1/ D. E. Farrar, The Investment Decision Under Uncertainty, (Englewood Cliffs, N. J.: Prentice Hall, Inc.), 1962, pp. 19-21.

2/ In general, this formulation gives the expected utility for a probability distribution over revenue in terms of the moments of the distribution and the derivatives of the utility function.

can, therefore, translate the function's origin and write

$$(9) \quad E[U(R)] = \mu + \frac{U''(\mu)}{2}\sigma^2$$

or as

$$(10) \quad E[U(R)] = \mu - A\sigma^2$$

where

$$(11) \quad A = -\frac{U''(\mu)}{2}.$$

For all 3 risk decision models, represented by equations (4), (5), and (6), the maximization of expected utility involves a choice among probability distributions of income and implies that an individual has a set of indifference curves between mean income (μ) and variance in income (σ^2) [i.e., a ceteris paribus increase in $\mu(\sigma)$ is desirable (undesirable)]. In each of the models, the individual is assumed to exhibit risk aversion and the slope of an indifference locus is positive (figure 2).[1/]

[1/] This property is demonstrated for the quadratic utility function in (5)

$$E[U(R)] = \int_{-\infty}^{\infty} [(1+b)R + bR^2]f(R)dR$$

$$= (1-b)\int_{-\infty}^{\infty} Rf(R)dR - b\int_{-\infty}^{\infty} R^2 dR$$

$$= (1-b)\,\mu - b\,(\sigma^2 + \mu^2).$$

Taking the total differential of $E[U(R)]$ and setting it to zero we have:
$dE[U(R)] = [(1-b) - 2b\mu]d\mu + [-2b\sigma]d\sigma = 0.$
Solving for the slope we have:

$$\frac{d\mu}{d\sigma} = \frac{\sigma}{-\frac{1+b}{2b} - \mu}$$

Recalling that we restricted $(1+b) + 2bR > 0$; and $b < 0$; then $-\frac{1+b}{2b} > 0$ and is necessarily larger than μ. Therefore, the slope of the indifference curve is positive.

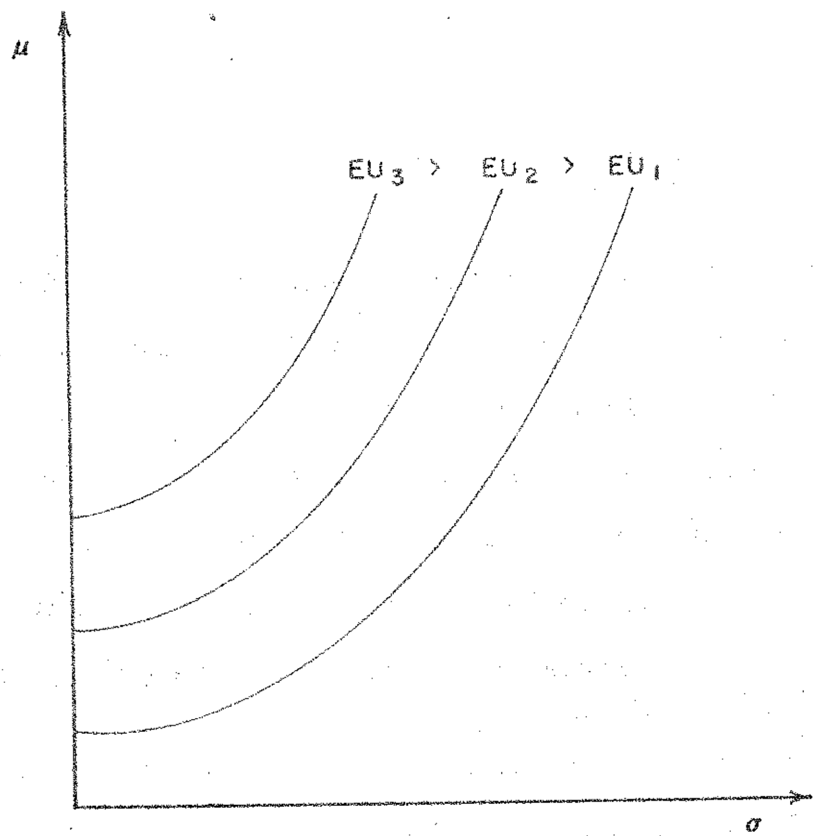

Figure 2. Risk Averter's Indifference Map

Although the form of the utility function for these decision models is determined, empirical implementation in each case does involve the selection of a value for the risk parameter. The parameters can be estimated by the methods described by Officer and Halter outlined above.[1] However, parameter values are estimated by these methods as specific to individual decision makers and the results may not generalize to other decision makers. In the event that estimating these parameter values is impractical or infeasible, one can select a wide range of parameter values and examine the model for each. This procedure allows one to approximate the decision maker's opportunity locus and provide him with valuable information about the mean income and variance in income associated with a variety of alternatives. The decision maker can then examine this information in light of his own attitude toward risk to choose from among the alternatives on the opportunity locus.[2]

So far the author has attempted to build an argument for and demonstrate the use of the expected utility maxim as a decision criterion in risk situations. The justification has been based on economic theory and the fact that decisions made according to the maxim are consistent with much observed behavior. The comparisons of other decision criteria with the expected utility maxim in the next two sections offer further evidence in support of the maxim.

1/ Officer and Halter, op. cit.

2/ Each point on the opportunity locus represents an alternative which yields a maximum expected income for a specific level of variance in income. The locus can be determined by parametric quadratic programming. It is demonstrated later that maximizing expected utility in the above three models involves choosing a point on the opportunity locus.

SAFETY-FIRST CRITERIA

While the development of decision criteria based on some concept of utility was taking place, another group of scientists were developing decision criteria based upon more objective concepts. These efforts were motivated largely by the arbitrary nature of utility functions. These criteria have become known as "safety-first" criteria.[1] Although these criteria differ from one another, they all have been motivated by the belief that a decision maker is not so much concerned with the possibility of small gains or losses but is very much concerned with being able to ward off total disaster.

The "Minimum α" Criterion

Economists have A. D. Roy to thank for the "minimum α" criterion.[2] This criterion was developed in response to Roy's belief that many people have a very real concept of what constitutes a personal economic disaster and that they believe the possibility of experiencing such a disaster exists. Consequently, he hypothesizes, people react by minimizing the probability of such a disaster. In other words, each individual has an income R^* which represents a minimum acceptable income. If his income falls below this level, the person believes he has experienced an economic disaster.

If a person has information concerning the expected income (μ) and the variance of income (σ^2) for all feasible choices open to him, he can minimize the probability of disaster in the following way.

1/ David H. Pyle and Stephen J. Turnovsky, "Safety-First and Expected Utility Maximization in Mean-Standard Deviation Portfolio Analysis," Review of Economics and Statistics, February 1970, p. 75.

2/ A. D. Roy, op. cit., pp. 430-449.

Appealing to Chebyshev's inequality we know that:[1]

$$(12)\quad P[|R-\mu| \geq \mu - R^*] \leq P(R \leq R^*) \leq \frac{\sigma^2}{(\mu - R^*)^2}\,.$$

For each possible action the above expression gives an upper bound on the probability of experiencing a disaster. To minimize this probability one need only choose that alternative which minimizes

$$(13)\quad \frac{\sigma^2}{(\mu - R^*)^2}\,.$$

The main theoretical advantage to using this decision criterion is that it introduces the concept of a disaster level of income. No one would argue that for some people preventing a disaster may be the primary motivation influencing their decisions, but, on the other hand, there may also be a large number of people who are not this conservative. However, the empirical application of this decision criterion may lead one to solutions which yield local maxims. These difficulties arise because the objective function $\frac{\mu - R^*}{\sigma^2}$ is not concave.[2]

The Maximum μ Criterion

Telser, in his attempt to salvage the "safety-first" approach to decision making from Roy's conservative assumptions, has proposed what can be called conveniently the "maximum μ criterion." The criterion can

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[1] For y, a random variable with expectation $E(y) = \mu$ and variance $V(y) = \sigma^2$, and $e > 0$, Chebyshev's inequality states that $P_r[Y - \mu \geq e] \leq \sigma^2/e^2$. See H. D. Brunk, An Introduction to Mathematical Statistics, (Waltham, Mass.: Blaisdell Publishing Company), second edition, 1965, p. 127 for a proof of the relationship.

[2] J. M. Boussard, "The Introduction of Risk into a Programming Model: Different Criteria and the Actual Behavior of Farmers," European Economic Review, Fall 1969, p. 98.

be described as follows:

> "Suppose that the entrepreneur does not want the probability of his net income falling short of R* to exceed α. Hence he will not choose any action such that $P(R \leq R^*;S) = P > \alpha$, [where S = action S]. This means all his actions can be put into one of two classes. The first class consists of all actions "S" such that $P(R \leq R^*;S) > \alpha$, and the second class consists of all actions S such that $P(R \leq R^*;S) \leq \alpha$. All the actions in the second class we shall call admissible. Then the entrepreneur will choose that action S of the admissible actions such that his expected income (μ) is maximum."[1/]

He goes on to show (also by the use of Chebyshev's inequality) that the objective function in applying this decision criterion is:

(14) Maximize μ subject to $\dfrac{\sigma^2}{(\mu - R^*)^2} \leq \alpha$.

Telser claims that his variation of the "safety-first" criterion is superior to Roy's version for two reasons. First, Telser is willing to assume that there is no risk to holding money; he confines his discussion to the short run to justify this assumption. Accordingly, Telser assumes that one can always avoid risk by holding cash, eliminating the need to minimize the probability of disaster. Second, he makes the point that adherence to Roy's criterion could, in extreme cases, lead to the choice of an action whose expected income was negative.

The obvious difficulty with this approach is that there are two decision parameters, α and R* which must be chosen before the decision rule can be applied. It is not clear that the choice of these two parameters may be made independently and there is nothing to suggest the exact nature of the dependence. There is also the possibility of selecting these parameters such that the admissible set is empty.

1/ Lester G. Telser, "Safety-First and Hedging," Review of Economic Studies, Vol. 23-1, 1955-57.

The "Maximum R*" Criterion

Shiniji Katoaka is responsible for a criterion which suggests that an entrepreneur wishes to assure himself some non-negative income with some specified high probability $(1 - \alpha)$.[1] Each alternative can guarantee him some income at this specified high probability level. Accordingly, he selects the portfolio which maximizes the income which he can be assured $[(1 - \alpha)100]$ percent of the time [i.e., max. R^* subject to $P(R \leq R^*) \leq \alpha$]. Appealing to Chebyshev's inequality once again, this criterion can be restated (for a given $\alpha = 1/k^2$) as

$$(15) \quad \max \mu - k\sigma.$$

The (E,L) Criterion

Baumol has suggested this criterion as an alternative to the Markowitz approach.[2] Although it is not classified as a "safety-first" criterion, it does suggest that the variance as a measure of risk is sometimes unreasonable. In this regard it is not unlike the "safety-first" criteria.

Baumol claims the (E,L) criterion eliminates the paradoxical cases which arise from the Markowitz approach. Just because an investment has a standard deviation (σ) one may not conclude a priori that it is not safe. If its expected value μ is high enough, the net result may be a high expected floor, $\mu - k\sigma$, beneath the future value of the investment (k is some constant). This paradox can be explained by the use of Baumol's example. Consider the two alternatives, A and B, represented in table 1.

1/ Shiniji Kataoka, "A Stochastic Programming Model," Econometrica, Vol. 31, January 1963.

2/ William J. Baumol, "An Expected Gain-Confidence Limit Criterion for Portfolio Selection," Management Science, Vol. 10, No. 1, October 1963, The Markowitz approach is based on the expected utility maxim and assumes a quadratic utility function.

Alternatives A and B are both "efficient" because neither dominates (A dominates C) the other (i.e., B has the larger expected return as well as a larger variance).

Table 1: Two Markowitz Efficient Alternatives

	A	B	C
μ	8	15	6
σ	2	4	2
$\mu+\sigma$	10	19	8
$\mu-\sigma$	6	11	4

Depending on the utility function, expected utility theory might recommend either A or B, but Baumol claims that if $\mu + \sigma$ and $\mu - \sigma$ are considered as the highest and lowest plausible outcomes, respectively, it may well be questioned whether anyone in his right mind would choose A. The worst anticipated outcome from choosing B is better than highest plausible outcome from A.

For people who make decisions on the basis of the (E,L) criterion, alternatives are compared on the basis of their expected values (μ) and the disaster level of income (R^*) which the alternatives imply. A person is assumed to be willing to sacrifice mean income only if he can expect an increase in the disaster level of income, $R^* = \mu - k\sigma$, to which he must be subjected. This comparison of alternatives on the basis of μ and R^* contrasts the μ and σ comparison of alternatives suggested by Markowitz.

The probability that net income will fall outside $\mu \pm k\sigma$ is a function of the value selected for k. According to Chebyshev's inequality, $P[\mu \pm k\sigma] \leq 1/k^2$. In other words, k is the risk parameter and as k increases the person becomes more conservative.

RELATIONSHIP BETWEEN EXPECTED UTILITY MAXIM AND "SAFETY-FIRST" CRITERIA

The exact nature of the relationships between the expected utility maxim and the "safety-first" criteria can be used to justify the use of the expected utility maxim. The purpose of this section is to examine the relationships.[1/]

These relationships will be derived assuming there is no riskless alternative available. To begin the comparison one must derive an investor's opportunity (efficiency) locus in the (μ, σ) space. This can be done by obtaining the alternative which minimizes σ for given levels of μ. Such a locus has the shape AB in figure 3.[2/] If one assumes

$$(16) \quad E(U) = V(\mu, \sigma) = \mu - k\sigma^2$$

where

$$\frac{\partial V}{\partial \mu} > 0; \quad \frac{\partial V}{\partial \sigma} < 0,$$

a set of indifference curves, $I_1 < I_2 < I_3$, can be superimposed on figure 3. Expected utility is maximized at point P where the decision maker's marginal rate of substitution of μ for σ equals the marginal rate of transformation of μ for σ along the opportunity locus.

Recalling that the objective for the "minimum α" criterion is to minimize $\frac{\sigma^2}{(\mu - R^*)^2}$, one can see that an indifference curve corresponding

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1/ D. H. Pyle and S. J. Turnovsky, op. cit., demonstrate these relationships for cases where distributions of net revenue can be fully described by two parameters. However, Chebyshev's inequality allows one to avoid the two parameter restriction.

2/ H. Markowitz, op. cit., one may also think of the process as finding the alternatives which maximize μ for given levels of σ.

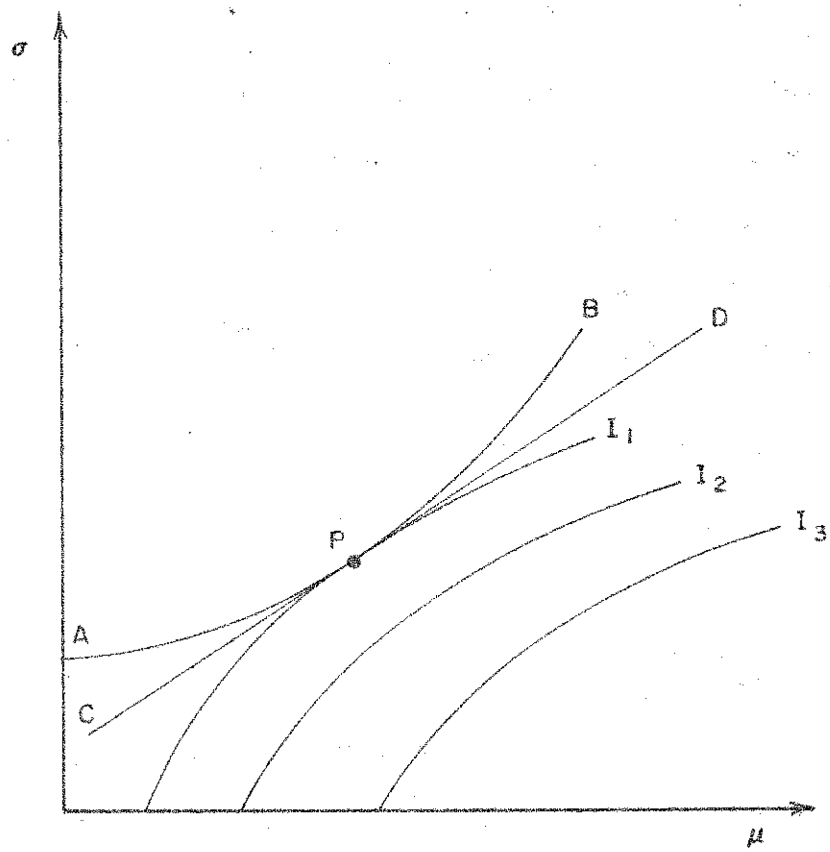

Figure 3. Expected Utility Maximization

to this criterion is given by

$$(17)\quad \frac{R^* - \mu}{\sigma} = C, \quad C < 0 \text{ (for small values of } \alpha \text{ which imply } R^* < \mu)$$

whose slope is positive

$$(18)\quad \frac{d\sigma}{d\mu} = -\frac{1}{C} = \frac{-\sigma}{\mu - R^*}\,.$$

It follows from (17) that all the indifference curves ($I_1' < I_2' < I_3'$) are linear and pass through a point on the μ axis (R^*,0) in figure 4.1/ A person who makes decisions according to this criterion will choose the alternative corresponding to point Q in figure 4.

The optimum alternative for the maximum μ criterion is not determined by any tangency conditions. Simple algebraic manipulation of (14) indicates that a person who makes decisions according to this criterion must choose from points in the set LN in figure 5. The alternative which maximizes μ subject to the constraint set LN is point N.

The set of indifference curves in the (μ,σ) plane for the "maximum R^*" criterion are similar to those of the "minimum α" criterion. From equation (15), one can see that a corresponding indifference curve is given by

$$(19)\quad \mu - k\sigma = C,$$

whose slope is

$$(20)\quad \frac{d\sigma}{d\mu} = \frac{1}{k}\,.$$

The set of indifference curves, represented in figure 6, is a set of parallel straight lines whose slope depends on $\alpha = 1/k^2$. The optimal alternative

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1/ This violates the law of transitivity. To avoid this difficulty one need only restrict attention to alternatives with positive variances.

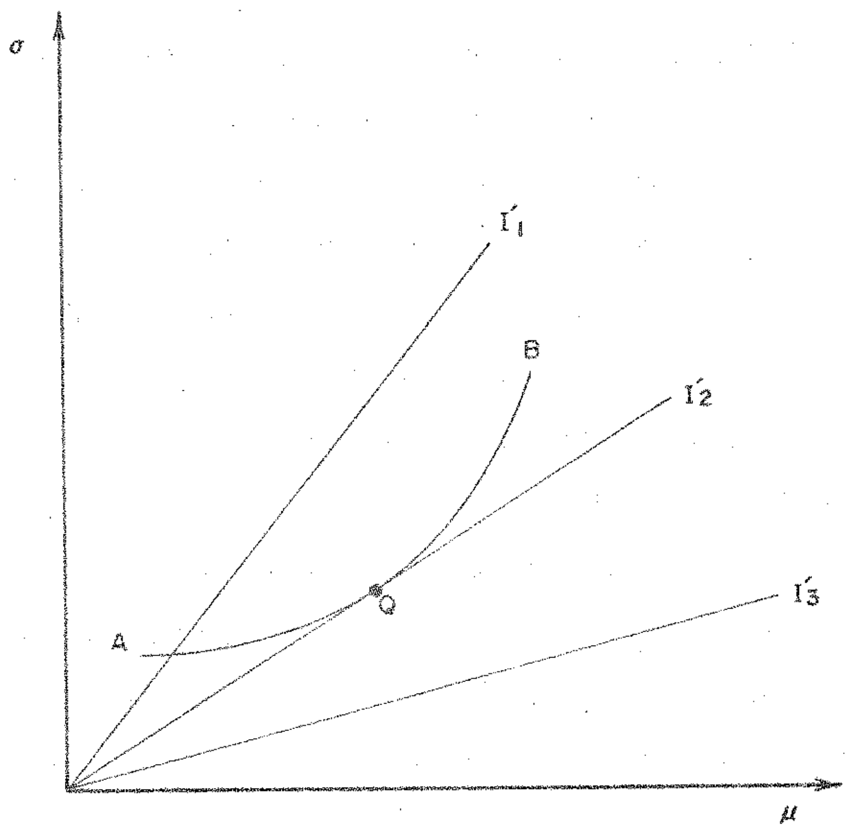

Figure 4. "Minimum α" Criterion

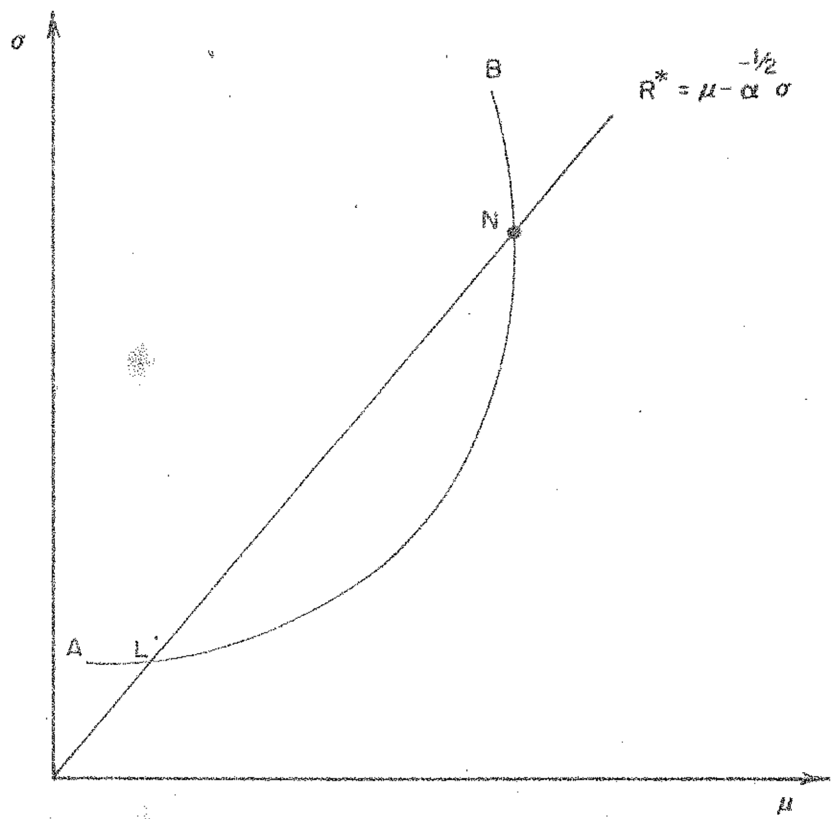

Figure 5. "Maximum μ" Criterion

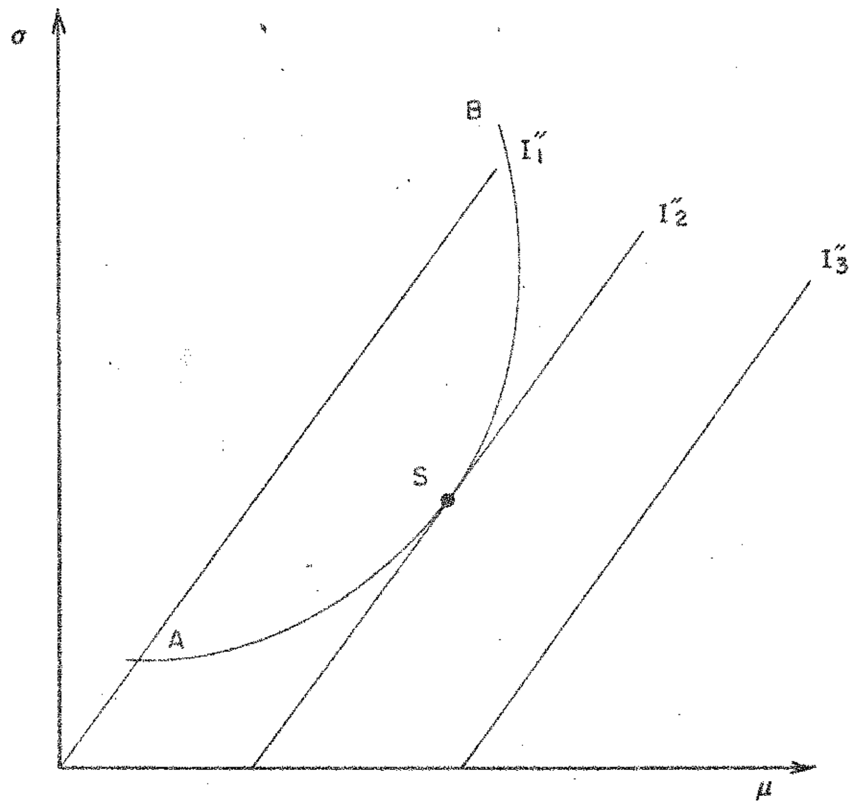

Figure 6. "Maximum R^*" Criterion

according to this criterion is given by point S.

The relationship between the expected utility maxim and each of the safety-first criteria is interesting and fortunate.

> "Referring to figure 3, imagine that an investor who maximizes utility chooses the (alternative) P. Since the set of attainable (alternatives) bounded by APB is a convex set and since the set of points preferable to those lying along I_1 also forms a convex set, by the separating hyper-plane theorem there exists a straight line CD passing through P which is tangential to both APB and I_1. If we use the "minimum α" criterion, we can equate the slope of this line to the slope given by (16) $\sigma/\mu - R^*$. For the given coordinates (μ_p, σ_p) of P, this determines a unique R^*, say $\hat{R}^*$. Thus, to the portfolio P chosen by an investor who maximizes expected utility, there corresponds a unique disaster level $\hat{R}^*$ such that the same portfolio would also be chosen by a safety-first investor who minizizes the probability of his total portfolio return falling below this disaster level."1/

Similarly, equating the slope of CD in figure 3 to $1/k$ determines a unique probability level, $\hat{\alpha} = 1/\hat{k}^2$, such that an investor who adopts the maximum R^* criterion will select the same portfolio as an expected utility maximizer.

This unique correspondence, unfortunately, does not hold when one tries to work back from these two safety-first criteria to the expected utility criterion. This can be verified by noting the fact that there is an infinity of utility functions which will lead expected utility maximizers to select any particular point on AB in figure 3.

Unlike the previous safety-first criteria, a person who operates according to the maximum μ criterion must choose two decision parameters. Therefore, the correspondence between this criterion and the expected utility maxim is not unique. This is easily verified by examining figure 5. There is an infinity of utility functions which would lead an expected utility maximizer to choose point N. Likewise, however, there is

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1/ D. H. Pyle and S. J. Turnovsky, op. cit., p. 78.

also an infinity of straight lines passing through point N, each of which is consistent with a person whose decisions are made according to the maximum μ criterion. The unique correspondence, similar to that mentioned above, becomes unique only after one of the two parameters is chosen.

The conclusions to be reached from these comparisons are quite simple. Under quite general conditions of risk-averse behavior, decisions recommended by expected utility theory and safety-first theory are operationally indistinguishable, although the psychological framework underlying each of these models is different. Furthermore, the comparisons argue in favor of adopting the expected utility model for empirical applications. Empirical results based on the application of the expected utility model can be used to make recommendations regarding the behavior of risk averters (i.e., expected utility maximizers). Such results can also be used directly to identify a particular safety-first investor who will act in a similar manner. In contrast, for any alternative selected by a safety-first investor, one can identify an infinity of expected utility investors who would make a similar selection. Identification of these utility functions requires a direct application of the expected utility maxim to the decision problem.

Similar conclusions are also true when applying the (E,L) criterion. The efficient set of alternatives corresponding to the (E,L) criterion is a proper subset of the efficient set corresponding to the expected utility criterion. That is, we know that there exists at least one utility function that will recommend each point on AB in figure 3. But, there are some points on AB that will never be recommended by the (E,L) criterion. Therefore, by assuming behavior consistent with expected utility maximization one could identify all alternatives consistent with the (E,L) criterion. The converse is not true.

EMPIRICAL APPRAISAL OF DECISION CRITERIA

So far the arguments supporting the expected utility maxim have taken the form of theoretical arguments based on comparisons of the expected utility maxim with other reasonable decision criteria. The expected utility maxim has been shown to have several advantages over the other criteria. Empirical evidence also suggests that application of the expected utility theory is appropriate.

Freund was one of the first to apply the expected utility criterion to farm planning in risk situations.[1/] His experience supports the hypothesis that farmers behave rationally. The production patterns suggested by his model were similar to those found on actual North Carolina farms. Recently, Boussard expanded Freund's analysis to include more complex decisions and arrived at similar conclusions.[2/] His study indicates that the expected utility criterion as well as the safety-first criteria perform quite well. They explain observed behavior but the results are very sensitive to the choice of the risk parameters.

In their efforts to estimate farmers' utility functions Officer and Halter uncover evidence indicating that farmers are basically risk averters.[3/] In addition quadratic utility functions fit the experimental observations very well. There was no significant improvement in specifying higher order utility functions.

Farrar's work in portfolio analysis also supports the hypothesis that investors faced with risk situations do behave rationally.[4/] The optimal

1/ R. J. Freund, op. cit.

2/ J. M. Boussard, op. cit.

3/ R. R. Officer and A. N. Halter, op. cit.

4/ D. E. Farrar, op. cit.

portfolios generated by his model resemble actual portfolios. However, not all the recent empirical evidence on portfolio analysis supports the use of expected utility theory based on mean income and variance in income. A recent study by M. J. Gordon examines investment behavior of the participants in a "portfolio game" and indicates that the participants demonstrated considerable aversion to risk. But, the findings are not consistent with a quadratic utility function.[1/] More specifically, the participants exhibited decreasing absolute and increasing relative risk aversion. The authors admit this evidence is suspect because of the way in which the data were generated. Their evidence cautions the reader not to generalize too much from the positive evidence supporting "quadratic" expected utility maximization. All the evidence is not in.

CONCLUSION

These empirical findings constitute the final evidence offered in support of the expected utility criterion in preference to some other decision criteria. Attention has been concentrated on those utility functions whose expectation is defined in terms of mean income and variance in income. Within these limits expected utility theory is appealing from a theoretical stand point because it is consistent with "rational" behavior. The theory is able to explain much observed behavior, particularly diversification and aversion to risk. Furthermore, information about behavior consistent with other accepted decision criteria is easily obtained from a direct application of the expected utility criteria.

Attention, in this paper, has been intentionally confined to the narrow subset of risk decision criteria which have been proposed. For

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1/ M. J. Gordon, et. al., "Experimental Evidence on Alternative Portfolio Decision Rules," American Economic Review, March 1972.

example, the study of expected utility theory has resulted in the development of utility functions other than the ones discussed earlier.[1] These were not considered because at the present time their empirical application is severely limited. Other risk decision criteria, such as Boussard's "focus loss" criteria, were not discussed, although it can be applied empirically. However, Boussard believes his use of "focus loss" sacrifices little generality and would surely arrive at a similar conclusion about the above discussion.[2]

1/ Ibid. Gordon lists several of the more common alternative utility functions.

2/ J. M. Boussard, "Time Horizon, Objective Function, and Uncertainty in a Multiperiod Model of Firm Growth," American Journal of Agricultural Economics, August 1971. "Focus loss" concentrates on the likeliness and importance of gains and losses. The focus loss is the minimum acceptable outcome and the focus gain is the corresponding value for gains. The criterion is similar to the safety-first criteria.

BIBLIOGRAPHY

Arrow, Kenneth J., "Aspects of the Theory of Risk-Bearing," Yrjo Jahnasson Lectures, Yrjo Jahnasson in Saatio, Helsinki, 1965.

Arrow, Kenneth J., "Alternative Approaches to the Theory of Choice in Risk Taking Situations," Econometrica, October 1951.

Baumol, W. J., "An Expected Gain-Confidence Limit Criterion for Portfolio Selection," Management Science, Vol. 10, No. 1, 1963.

Bernoulli, D., "Exposition of a New Theory on the Measurement of Risk," Papers of the Imperial Academy of Science, V. Petersburg, 1738, trans. L. Sommer, Econometrica, January 1954.

Boussard, J. M., "The Introduction of Risk into a Programming Model: Different Criteria and the Actual Behavior of Farmers," European Economic Review, Fall 1969.

Charnes, A. and W. W. Cooper, "Chance Constrained Programming," Management Science, Vol. 6, 1959.

Cocks, K. D., "Discrete Stochastic Programming," Management Science, Vol. 15, No. 1, 1968.

Farrar, Donald E., The Investment Decision Under Uncertainty, Englewood Cliffs, New Jersey: Prentice Hall, Inc., 1962.

Freund, R. J., "The Introduction of Risk into a Programming Model," Econometrica, Vol. 21, No. 4, October 1956.

Friedman, M. and L. J. Savage, "The Utility Analysis of Choices Involving Risk," in Readings in Price Theory, ed. American Economic Association, Chicago: Richard D. Irwin, Inc., 1952.

Halter, A. N. and G. W. Dean, Decisions Under Uncertainty with Research Applications, Chicago: South-Western Publishing Co., 1971.

Katoaka, Shiniji, "A Stochastic Programming Model," Econometrica, Vol. 31, January 1963.

Latanė, H. A., "Criteria for Choice Among Risky Ventures," Journal of Political Economy, April 1959.

Luce, R. D. and H. Raiffa, Games and Decisions, New York: John Wiley and Sons, Inc., 1957.

Markowitz, Harry, Portfolio Selection: Efficient Diversification of Investments, New York: John Wiley and Sons, Inc., 1959.

Markowitz, Harry, "Portfolio Selection," The Journal of Finance, Vol. 7, No. 1, March 1952.

Naslund, Bertil, "Mathematical Programming Under Risk," The Swedish Journal of Economics, Vol. LXVII, No. 3, 1965.

Officer, R. R. and A. N. Halter, "Utility Analysis in a Practical Setting," American Journal of Agricultural Economics, May 1968.

Pyle, David H. and S. J. Turnovsky, "Safety-First and Expected Utility Maximization in Mean-Standard Deviation Portfolio Analysis," The Review of Economics and Statistics, February 1970.

Roy, A. D., "Safety-First and the Holding of Assets," Econometrica, 20, July 1952.

Shackle, G. L. S., Expectation in Economics," in Uncertainty and Business Decisions: A Symposium, ed. C. Carter, G. Meredith, and G. Shackle, Liverpool: The University Press, 1954.

Sharpe, W. F., "A Simplified Model for Portfolio Analysis," Management Science, Vol. 10, No. 2, 1963.

Sharpe, W. F., Portfolio Analysis Based on a Simplified Model of the Relationships Among Securities, unpublished doctoral dissertation, Los Angeles: University of California at Los Angeles.

Telser, L., "Safety-First and Hedging," Review of Economic Studies, 23, 1955-56.

Tobin, James, "Liquidity Preference as Behavior Toward Risk," Review of Economic Studies, February 1958.

Von Neumann, J. and O. Morgenstern, Theory of Games and Economic Behavior, Princeton, N. J.: Princeton University Press, 1947.

Wallingford, B. A., "A Survey and Comparison of Portfolio Selection Models," Journal of Financial and Quantitative Analysis, June 1967.